Better-quasi-order: ideals and spaces

Ainsi de suite...

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To Lolita

Bien que les pieds de l'homme
n'occupent qu'un petit coin de la terre,
c'est par tout l'espace qu'il n'occupe pas
que l'homme peut marcher sur la terre immense.

Bien que l'intelligence de l'homme
ne pénètre qu'une parcelle de la vérité totale,
c'est par ce qu'elle ne pénètre pas
que l'homme peut comprendre ce qu'est le ciel.

— *Tchouang-tseu*

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1 Introduction

Mathematicians have imagined a myriad of objects, most of them infinite, and inevitably followed by an infinite suite.

What does it mean to understand them? How does a mathematician venture to make sense of these infinities he has imagined?

Perhaps, one attempt could be to organise them, to arrange them, to order them. At first, the mathematician can try to achieve this in a relative sense by comparing the objects according to some idea of complexity; this object should be above that other one, those two should be side by side, etc. So the graph theorist may consider the *minor relation* between graphs, the recursion theorist may study the *Turing reducibility* between sets of natural numbers, the descriptive set theorist can observe subsets of the Baire space through the lens of the *Wadge reducibility* or equivalence relations through the prism of the *Borel reducibility*, or the set theorist can organise ultrafilters according to the *Rudin-Keisler ordering*.

This act of organising objects amounts to considering an instance of the very general mathematical notion of a *quasi-order* (qo), namely a transitive and reflexive relation.

As a means of classifying a family of objects, the following property of a quasi-order is usually desired: a quasi-order is said to be *well-founded* if every non-empty sub-family of objects admits a minimal element. This means that there are minimal – or simplest – objects which we can display on a first bookshelf, and then, amongst the remaining objects there are again simplest objects which we can display on a second bookshelf above the previous one, and so on and so forth – most probably into the transfinite.

However, as a matter of fact another concept has been 'frequently discovered' [Kru72] and proved even more relevant in diverse contexts: a *well-quasi-order* (WQO) is a well-founded quasi-order which contains no infinite antichain. Intuitively a well-quasi-order provides a satisfactory notion of hierarchy: as a well-founded quasi-order, it comes naturally equipped with an ordinal rank and there are up to equivalence only finitely many elements of any given rank. To prolong our metaphor, this means that, in addition, every bookshelf displays only finitely many objects – up to equivalence.

The theory of WQOs consists essentially of developing tools in order to show that certain quasi-orders suspected to be WQO are indeed so. This theory exhibits a curious and interesting phenomenon: to prove that a certain quasi-order is WQO, it may very well be easier to show that it enjoys a much stronger property. This observation may be seen as a motivation for considering the complicated but ingenious concept of *better-quasi-order* (BQO) invented by Crispin St. J. A. Nash-Williams [Nas65]. The concept of BQO is weaker than that of well-ordered set but it is stronger than that of WQO. In a sense, WQO is defined by a single 'condition', while uncountably many 'conditions' are necessary to characterise BQO. Still, as Joseph B. Kruskal [Kru72, p.302] observed in 1972: 'all "naturally occurring" WQO sets which are known are BQO'[1].

The first contribution of this thesis is to the theory of WQO and BQO. The main result is the proof of a conjecture made by Maurice Pouzet [Pou78] which states that any WQO whose ideal completion remainder is BQO is actually BQO. Our proof relies on new results with both a combinatorial and a topological flavour concerning maps from a *front* into a compact metric space. We think that these results are of independent interest and hope that they can be applied in other situations where fronts and barriers are used, as in the theory of Banach spaces for example.

Our second contribution is of a more applied nature and deals with topological spaces. We define a quasi-order on the subsets of every second countable T_0 topological space in a way that generalises the Wadge quasi-order on the Baire space, while extending its nice properties to virtually all these topological spaces.

Our starting point is the celebrated Wadge quasi-order – of reducibility by continuous functions – on subsets of the Baire space. This quasi-order is described by Alessandro Andretta and Alain Louveau [AL] as 'the ultimate analysis of the subsets of the Baire space'. The fact that the extremely fine Wadge quasi-order is WQO on Borel sets is doubtless among the most attractive of its properties. The proof of this fact given by Tony Martin, building on previous work by Leonard Monk, is an example of one of the main techniques of BQO theory, namely the use of infinite games and determinacy. Moreover, as we explain in this thesis, this property of the Wadge quasi-order follows from an extension of the idea underlying the very definition of a BQO.

For other important topological spaces the quasi-order of reducibility by continuous functions is however far less satisfactory. For example, the family of Borel subsets of the real line is very far from being WQO under continuous reducibility. While reducibility by discontinuous functions has been studied by

[1] The minor relations on finite graphs, proved to be WQO by Robertson and Seymour [RS04], is to our knowledge the only naturally occurring WQO which is not *yet* known to be BQO.

some authors to remedy this situation, we propose instead to keep continuity but to weaken the notion of function to that of relation. Using the notion of *admissible representation* studied in *Type-*2 *theory of effectivity*, we define the quasi-order of reducibility by relatively continuous relations. We show that this quasi-order both refines the classical hierarchies of complexity and is WQO on the Borel subsets of virtually every second countable T_0 space.

1.1 From well to better

A quasi-order Q is a WQO if it contains no infinite descending chain nor infinite antichain. However using Ramsey's Theorem this is equivalent to the absence of a so-called *bad sequence*, namely a sequence $(q_n)_{n\in\omega}$ such that $m < n$ in ω implies $q_m \nleq q_n$ in Q.

The concept of *better-quasi-order* was invented by Nash-Williams [Nas65]. Its definition relies on a generalisation of Ramsey's Theorem to transfinite dimension: the notion of *front*. It generalises the definition of WQO given above in the sense that it does not only forbid bad sequences, but also bad sequences of sequences, bad sequences of sequences of sequences and so on and so forth in the transfinite. A front can be thought of as a convenient notion of index sets for these sequences of … of sequences and we call any map from a front into some set a *super-sequence*. A BQO is then a quasi-order which admits no bad super-sequence.

One contribution of this thesis is to show that super-sequences deserve their name since they share significant properties with usual sequences. A crucial property for a sequence in the context of metric spaces is the Cauchy condition. In order to generalise the notion of being Cauchy to super-sequences, we observe that a sequence $(x_n)_{n\in\omega}$ in a metric space $\mathcal{X}$ satisfies the Cauchy condition if and only if the mapping $\omega \to \mathcal{X}$, $n \mapsto x_n$ is uniformly continuous, when ω is identified with a subspace of the Cantor space 2^ω via $n \mapsto 0^n 10^\omega$.

As observed notably by Todorčević [AT05; Tod10], fronts can naturally be seen as subsets of the Cantor space. Being a compact Hausdorff space, the Cantor space admits a unique uniformity that is compatible with its topology. Even though a front is a discrete topological subspace of 2^ω, we observe that it inherits a non-trivial uniformity from 2^ω. Let us say that a super-sequence in a metric space is Cauchy when it is uniformly continuous. We show the following theorem, which generalises the usual sequential compactness for metric spaces.

Theorem 1.1 (with R. Carroy)**.** *Every super-sequence in a compact metric space has a Cauchy sub-super-sequence.*

This combinatorial result should be compared with Erdös-Rado Theorem [ER50] and Pudlak-Rödl Theorem [PR82] as a Ramsey theorem for partitions into infinitely many classes. We also note also that this result subsumes Nash-Williams' Theorem.

Given a complete metric space $\mathcal{X}$, every Cauchy sequence $f : \omega \to \mathcal{X}$ converges, and thus extends to a continuous map $\overline{f} : \overline{\omega} \to \mathcal{X}$, where $\overline{\omega}$ is the one point compactification of ω. The same is true about Cauchy super-sequences: any uniformly continuous super-sequence $f : F \to \mathcal{X}$ from a front F into a complete metric space $\mathcal{X}$ continuously extends to the uniform completion $\overline{F}$ of F, which coincides with the topological closure of the front inside the Cantor space, to yield a continuous map $\overline{f} : \overline{F} \to X$.

We also study the continuous extension of Cauchy super-sequences. In full generality, we are concerned with continuous maps from the topological closure of a front into some topological space.

Recall that a point x in a topological space $\mathcal{X}$ is called *isolated* if the singleton $\{x\}$ is open in $\mathcal{X}$, and *limit* otherwise. The following simple fact exhibits a property of converging sequences that can always be achieved by going to a subsequence: If $(x_n)_{n\in\omega}$ is a sequence converging in a topological space $\mathcal{X}$ to some point x, then there is a subsequence $(x_n)_{n\in N}$ such that

1. if x is isolated, then $(x_n)_{n\in N}$ is constant equal to x;
2. if x is limit, then

   either x_n is isolated for all $n \in N$;

   or x_n is limit for all $n \in N$.

We generalise this fact to super-sequences by notably showing the following result.

Theorem 1.2 (with R. Carroy)**.** *Let $\overline{f} : \overline{F} \to \mathcal{X}$ be a continuous extension of a super-sequence f in a topological space $\mathcal{X}$. Then there exists a sub-super-sequence $f' : F' \to \mathcal{X}$ of f such that*

either *$\overline{f'} : \overline{F'} \to X$ is constant and equal to an isolated point;*

or *$\{s \in \overline{F'} \mid f(s) \text{ is limit}\} = \overline{G}$ for some front G.*

We then apply these theorems to the theory of BQO. The main result is a proof of a conjecture made by Pouzet [Pou78] in his *thèse d'état*. By an *ideal* of a quasi-order Q we mean a downward closed and up-directed subset of Q. To every element $q \in Q$ corresponds the *principal ideal* $\downarrow q = \{p \in Q \mid p \leqslant q\}$.

The *ideal completion* of Q is defined as the set of ideals of Q partially ordered by inclusion, and it is denoted by $\mathrm{Id}(Q)$. Notice that Q embeds into $\mathrm{Id}(Q)$ via the map $e : q \mapsto \downarrow q$. We denote by $\mathrm{Id}^*(Q)$ the set $\mathrm{Id}(Q) \setminus e(Q)$ of non-principal ideals of Q, this is the remainder of the ideal completion of Q.

The statement of the conjecture made by Pouzet [Pou78] is the following:

Theorem 1.3 (with R. Carroy)**.** *If Q is* WQO *and* $\mathrm{Id}^*(Q)$ *is* BQO*, then Q is* BQO*.*

Pouzet and Sauer [PS06] advanced a proof of this statement, but their proof contains a gap, as clearly revealed by Alberto Marcone and acknowledged by Pouzet and Sauer. While the approach of Pouzet and Sauer [PS06] is purely combinatorial, we follow a completely different line and make essential use of the fact that the ideal completion of WQO admits a natural compact topology.

We view the importance of the ideal completion as the coincidence in the case of a WQO of several notions of completions of a quasi-order. In fact, gathering many results and facts which certainly belong to the folklore we obtain the following:

Theorem 1.4. *For a* WQO *P the following completions coincide:*

(i) *the ideal completion* $\mathrm{Id}(P)$ *equipped with the Lawson topology,*

(ii) *the Cauchy ideal completion of P,*

(iii) *the Nachbin order-compactification, or ordered Stone-Čech compactification, of P with the discrete topology.*

The different properties of these three different completions combine to give what we call the *ideal space* of a WQO. This enables us to show that Theorem 1.1 admits the following nice corollary in the context of WQO theory.

Theorem 1.5 (with R. Carroy)**.** *Every super-sequence $f : F \to Q$ into a* WQO *Q admits a Cauchy sub-super-sequence $f' : F' \to Q$, which therefore extends to a continuous map $\overline{f'} : \overline{F'} \to \mathrm{Id}(Q)$ into the ideal space of Q.*

As a matter of fact, the ideal space of a WQO is a scattered compact space whose limit points are exactly the non principal ideals. Applying Theorem 1.2, this allows us to prove that any bad super-sequence in a WQO Q yields a bad super-sequence into the non principal ideals of Q. Therefore proving Pouzet's conjecture.

1.2 A well-quasi-order on the subsets of a topological space

The versatile concept of a topological space has proved valuable in various areas of mathematics. In many cases of interest, the spaces are second countable, i.e. their topology admits a countable base. While separable metrisable spaces are of primary importance to Analysis [Kec95], topological spaces that do not satisfy the Hausdorff separation property are central to Algebraic Geometry [EH00] and to Computer Science [Gou13]. We consider without distinction all second countable spaces which satisfy the weakest separation property T_0, namely every two points which have exactly the same neighbourhoods are equal.

We are interested in finding a way to quasi-order the subsets of a topological space according to their complexity. Among the properties of such a quasi-ordering the following are arguably desired.

- It should agree with an a priori idea of topological complexity, in particular it should refine the classical hierarchies.
- It should be as fine as possible.
- It should be WQO or even BQO – at least on Borel subsets.

The very act of defining a topology on a set of objects consists in specifying simple, easily observable properties: the open sets. We are then interested in understanding the complexity of the other subsets relatively to the open sets.

Already at the turn of the twentieth century, the French analysts – Baire, Borel and Lebesgue – stratified the Borel sets of a metric space into a transfinite hierarchy: the Baire classes $\boldsymbol{\Sigma}^0_\alpha$, $\boldsymbol{\Pi}^0_\alpha$ and $\boldsymbol{\Delta}^0_\alpha$. These classes are well-known to exhibit the following pattern:

$$
\begin{array}{ccccccccccccccccc}
\boldsymbol{\Sigma}^0_1 & \subsetneq & & \subsetneq & \boldsymbol{\Sigma}^0_2 & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & \boldsymbol{\Sigma}^0_\alpha & \subsetneq & & \subsetneq & \\
 & & \boldsymbol{\Delta}^0_2 & & & & \boldsymbol{\Delta}^0_3 & & \cdots & & \boldsymbol{\Delta}^0_\alpha & & & & \boldsymbol{\Delta}^0_{\alpha+1} & & \cdots \\
\boldsymbol{\Pi}^0_1 & \subsetneq & & \subsetneq & \boldsymbol{\Pi}^0_2 & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & \boldsymbol{\Pi}^0_\alpha & \subsetneq & & \subsetneq &
\end{array}
$$

Borel sets are thus classified according to the complexity of their definition from open sets along this transfinite ladder. This classification was further refined by Hausdorff, and later by Kuratowski, by identifying what is now called the difference hierarchies, consisting of the Hausdorff–Kuratowski classes $D_\xi(\boldsymbol{\Sigma}^0_\alpha)$. Since for every map $f : \mathcal{X} \to \mathcal{X}$, the preimage function $f^{-1} : \mathcal{P}(\mathcal{X}) \to$

$\mathcal{P}(\mathcal{X})$ is a complete Boolean homomorphism, it directly follows from their definition that the Borel classes and the Hausdorff–Kuratowski classes are *closed under continuous preimages*[2].

Wadge in his Ph.D. thesis [Wad82] was the first to investigate the quasi-order of *continuous reducibility* on the subsets of the Baire space ω^ω: for $A, B \subseteq \omega^\omega$ we say that A is *Wadge reducible* to B, $A \leqslant_{\mathrm{W}} B$, if and only if there exists a continuous $f : \omega^\omega \to \omega^\omega$ such that $f^{-1}(B) = A$. This quasi-order called the *Wadge quasi-order* relates to the complexity of the subsets of the Baire space in the sense that $A \leqslant_{\mathrm{W}} B$ if and only if one can continuously reduce the membership problem for A to the membership problem for B. This quasi-order is remarkable. By considering suitable infinite games and using the determinacy of these games, which follows from Borel determinacy, this quasi-order turns out to be well-founded and to admit antichains of size at most 2 on the Borel sets. As Andretta and Louveau [AL] describe in their introduction to [KLS12]: 'The Wadge hierarchy is the ultimate analysis of $\mathcal{P}(\omega^\omega)$ in terms of topological complexity'. While the Borel classes and the Hausdorff–Kuratowski classes are closed under continuous preimages, and therefore represent initial segments for $\leqslant_{\mathrm{W}}$, there are in fact many more initial segments, so that the Wadge qo refines greatly these classical hierarchies.

Of course the *quasi-order of continuous reducibility* can be defined in any topological space $\mathcal{X}$ in the obvious way, for $A, B \subseteq \mathcal{X}$ let $A \leqslant_{\mathrm{W}} B$ if and only if there exists a continuous function $f : \mathcal{X} \to \mathcal{X}$ such that $A = f^{-1}(B)$. The nice properties of the Wadge quasi-order extend easily to all zero-dimensional Polish spaces, or even to all Luzin – or Borel absolute – zero-dimensional spaces. The restriction to Luzin spaces and their Borel subsets comes from the use of determinacy in the proof, but it can be weaken if one is willing to assume the determinacy of a larger class of games. In particular assuming the Axiom of Determinacy, the same holds for the quasi-order of continuous reducibility on all subsets of any zero-dimensional second countable space.

However the restriction to zero-dimensional spaces is of a different nature. In fact when the space is not zero-dimensional there may be very few continuous functions, independently of any determinacy hypothesis. Hertling in his Ph.D. thesis [Her96] shows that the qo of continuous reducibility of the Borel subsets of the real line $\mathbb{R}$ exhibits a more complicated pattern than in the case of the Baire space. For example, Ikegami showed in his Ph.D. thesis [Ike10] (see also [IST]) that the powerset $\mathcal{P}(\omega)$ of ω partially ordered by inclusion modulo finite – and hence any partial order of size $\aleph_1$ – embeds in the qo of continuous

[2] i.e. for every $A \subseteq \mathcal{X}$ in the class and every continuous $f : \mathcal{X} \to \mathcal{X}$ the set $f^{-1}(A)$ belongs to the class.

reducibility of Borel sets of the real line (cf. Subsection 5.6.1). In a more general setting, Schlicht showed in [Sch] that in any non zero-dimensional metric space there is an antichain for the qo of continuous reducibility of size continuum consisting of Borel sets. Selivanov [Sel06, and references there] and also Becher and Grigorieff [BG15] studied continuous reducibility in non Hausdorff spaces, where the situation is in general much less satisfactory than in the case of the Baire space.

In search for a useful notion of hierarchy outside Polish zero-dimensional spaces, Motto Ros, Schlicht, and Selivanov [MSS15] consider reductions by discontinuous functions. For example they obtain that the Borel subsets of the real line are well-founded with antichains of size at most 2 when quasi-ordered by reducibility via functions $f : \mathbb{R} \to \mathbb{R}$ such that for every $A \in \mathbf{\Sigma}^0_3(\mathbb{R})$ we have $f^{-1}(A) \in \mathbf{\Sigma}^0_3$. They leave open the question whether $\mathbf{\Sigma}^0_3$ can be replaced by $\mathbf{\Sigma}^0_2$ in the above statement. Arguably one defect of this qo is that it does not refine the low level Borel classes, nor does it respect the Hausdorff hierarchy of the $\mathbf{\Delta}^0_2$.

Instead of considering reduction by discontinuous functions, we propose to keep continuity but to release the second concept at stake, namely that of function. In the abstract, our first remark is that *total relations* account perfectly for the idea of reducibility and in fact generalise the framework of reductions as functions.

The notion of continuity for relations that fits our purpose is called *relative continuity*. It relies on the simple and fundamental concept of *admissible representation* of a topological space which is the starting point of the development of computable analysis from the point of view of Type-2 theory of effectivity [Wei00].

The basic idea is to represent the points of a topological space $\mathcal{X}$ by means of infinite sequences of natural numbers. Given such a representation of $\mathcal{X}$, i.e. a partial surjective function $\rho :\subseteq \omega^\omega \to \mathcal{X}$, an $\alpha \in \omega^\omega$ is a *name* for a point $x \in \mathcal{X}$ when $\rho(\alpha) = x$. A function $f : \mathcal{X} \to \mathcal{X}$ is then said to be *relatively continuous* (resp. computable) with respect to ρ if the function f is continuous (or computable) in the ρ-names, i.e. there exists a continuous (resp. computable) $F : \operatorname{dom}\rho \to \operatorname{dom}\rho$ such that $f \circ \rho = \rho \circ F$. Of course the notion of relatively continuous function depends on the considered representation. However, for every second countable T_0 space $\mathcal{X}$ there exists – up to equivalence – a greatest representation (cf. Theorem 5.17) among the continuous ones, called an *admissible representation* of $\mathcal{X}$. The importance of admissible representations resides in the following fact (cf. Theorem 5.24): for an admissible representation ρ of $\mathcal{X}$, a function $f : \mathcal{X} \to \mathcal{X}$ is relatively continuous with respect to ρ if and only if f is continuous. Notice however that in general as long as the representation is *not* injective, many continuous transformations of the names exist which do not induce a map on the space $\mathcal{X}$. Indeed different names α, β of some point x can be sent by a continuous function F onto names $F(\alpha)$, $F(\beta)$ representing different points, i.e. $\rho(F(\alpha)) \neq \rho(F(\beta))$. Such transformations are called *relatively continuous relations* (cf. Definition 5.29) and they were first investigated in a systematic manner by Brattka and Hertling [BH94].

We propose to consider reducibility by total relatively continuous relations. When we fix an admissible representation ρ of a second countable T_0 space $\mathcal{X}$, it is natural to think of reductions by relatively continuous relations as 'reductions in the names': if $A, B \subseteq \mathcal{X}$, then A reduces to B, in symbols $A \preccurlyeq_{\mathrm{W}} B$, if and only if there exists a continuous function F from the names to the names such that for every name α, $\rho(\alpha) \in A \leftrightarrow \rho(F(\alpha)) \in B$. In other words, for every point x and every name α for x, $F(\alpha)$ is the name of a point that belongs to B if and only if x belongs to A.

We wish to mention that in 1981 Tang [Tan81] worked with an admissible representation of the Scott domain $\mathcal{P}\omega$ and studied on this particular space the exact same notion of reduction that we propose here in a more general setting. But firstly, this study is antecedent to the introduction by Kreitz and Weihrauch [KW85] of the concept of admissible representation and Tang does not notice that his representation of $\mathcal{P}\omega$ is admissible. This remark is indeed important since it allows one to see that his notion of reduction is actually topological, namely it depends only on the topology of the space $\mathcal{P}\omega$. Secondly, even though his paper is often cited, no author seem to notice his particular approach to reducibility on $\mathcal{P}\omega$.

To confront the quasi-order $\preccurlyeq_{\mathrm{W}}$ of reducibility by relatively continuous relations to our expectations, we show the following results.

Firstly, we show that reducibility by relatively continuous relations is a generalisation of Wadge reducibility outside zero-dimensional spaces.

Proposition 1.6. *On every zero-dimensional space, the reducibility by relatively continuous relations coincides with the continuous reducibility.*

Notice however that using a result of Schlicht [Sch] we show that it differs from the continuous reducibility in every separable metrisable space that is not zero-dimensional.

Secondly, using a result by Saint Raymond [Sai07] extended by de Brecht [deB13] we obtain that that reducibility by relatively continuous relations refines the classical hierarchies of Borel and Hausdorff–Kuratowski.

Proposition 1.7. *Let $\mathcal{X}$ be a second countable T_0 spaces and A and B be subsets of $\mathcal{X}$. For every $1 \leqslant \alpha, \xi < \omega_1$,*

(i) *if* $B \in \mathbf{\Sigma}^0_\alpha(\mathcal{X})$ *and* $A \preccurlyeq_W B$*, then* $A \in \mathbf{\Sigma}^0_\alpha(\mathcal{X})$*,*

(ii) *if* $B \in D_\xi(\mathbf{\Sigma}^0_\alpha(\mathcal{X}))$ *and* $A \preccurlyeq_W B$*, then* $A \in D_\xi(\mathbf{\Sigma}^0_\alpha(\mathcal{X}))$*.*

Finally, we show that the quasi-order $\preccurlyeq_{\mathrm{W}}$ is as well behaved on the Borel sets of a very large class of second countable T_0 spaces as the Wadge quasi-order is on the Borel subsets of the Baire space. The use of Borel determinacy naturally leads us to define the class of *Borel representable spaces*, which contains every Borel subspace of the Scott domain $\mathcal{P}\omega$, and in particular every Borel subspace of a Polish space.

Theorem 1.8. *Let* $\mathcal{X}$ *be a Borel representable space. Then the reducibility by relatively continuous relations* $\preccurlyeq_W$ *is well-founded on the Borel subsets of* $\mathcal{X}$*. Moreover the Wadge Lemma holds, namely for every Borel subset* A *and* B *of* $\mathcal{X}$

$$\text{either } A \preccurlyeq_W B \text{ or } B \preccurlyeq_W A^{\complement}.$$

As in the case of the Baire space, this structural result depends on the determinacy of the games under consideration. In particular, under the Axiom of Determinacy, the above theorem extends to all subsets of every second countable T_0 space.

1.3 Organisation of the thesis

Chapter 2: Sequences in sets and orders Several articles – notably [Mil85; Kru72; Sim85; Lav71; Lav76; For03] – contains valuable introductory material to the theory of better-quasi-orders. However, a book entitled 'Introduction to better-quasi-order theory' is yet to be written. This chapter represents our attempt to give the motivated introduction to the deep definition of Nash-Williams we wished we had when we began studying the theory two years before.

In Section 2.1 we prove a large number of characterisations of well-quasi-orders, all of them are folklore except the one stated in Proposition 2.14 which benefits from both an order-theoretical and a topological flavour.

We make our way towards the definition of better-quasi-orders in Section 2.2. One of the difficulties we encountered when we began studying better-quasi-order is due to the existence of two main different definitions – obviously equivalent to experts – and along with them two different communities, the graph theorists and the descriptive set theorists, who only rarely cite each other in their contributions to the theory. The link between the original approach of

Nash-Williams (graph theoretic) with that of Simpson (descriptive set theoretic) is merely mentioned by Argyros and Todorčević [AT05] alone. We present basic observations in order to remedy this situation in Subsection 2.2.3. Building on an idea due to Forster [For03], we introduce the definition of better-quasi-order in a new way, using insight from one of the great contributions of descriptive set theory to better-quasi-order theory, namely the use of games and determinacy.

Finally in Section 2.3 we put the definition of better-quasi-order into perspective. This last section contains original material which have not yet been published by the author.

Chapter 3: Sequences in spaces Building on the previous chapter, we study super-sequences in metric spaces. After making some simple observations on Cauchy sequences, we define Cauchy super-sequences in Section 3.1 and collect some basic facts about the closure of a front inside 2^ω. The main result of this chapter is that any super-sequence into a compact metric space $\mathcal{X}$ admits a Cauchy sub-super-sequence. This general result actually follows easily from the particular case where $\mathcal{X}$ is the Cantor space. Our reason to focus on the Cantor space lies in the fact that uniform continuity admits of a nice characterisation in the zero-dimensional setting as showed in Section 3.2. In particular the uniform structure of a front F essentially consists in a distinguished countable Boolean algebra of subsets of F (a characterisation of which is given in Proposition 3.20), that we call the blocks of the front.

In Section 3.3 we prove that any countable family of subsets of a front can be turned into blocks by eventually going to a sub-front in Theorem 3.24. From this combinatorial result we deduce that every super-sequence in 2^ω admits a Cauchy super-sequence.

Of course, when $\mathcal{X}$ is a complete metric space and $f : F \to \mathcal{X}$ is a Cauchy super-sequence, then f extends to a continuous map $\bar{f} : \overline{F} \to \mathcal{X}$, where $\overline{F}$ is the topological closure of F inside 2^ω. We study continuous map from the closure $\overline{F}$ of a front into an arbitrary topological space in Section 3.4.

In particular we define a certain 'normal form' for the continuous functions $f : \overline{F} \to \mathcal{Y}$ from the closure of a front F into a topological space $\mathcal{Y}$ and we prove that this normal form can always be achieved by a restriction to some $\overline{H}$ where H is a sub-front of F.

Importantly, these results are applied in the next chapter to prove the conjecture suggested by Pouzet [Pou78].

While the exposition given in this chapter is new, the results are published in an article [CP14] by the author and R. Carroy in *Fundamenta Mathematicae*.

Chapter 4: The ideal space of a well-quasi-order The main result of this chapter is the proof of a conjecture made by Pouzet [Pou78] which relates the BQO character of a given WQO with the BQO character of the remainder of the ideal completion of the WQO. Our approach relies on the fact that the ideal completion of a WQO is actually a compactification. This can be explained by the coincidence in the case of a WQO of the ideal completion with two other important completions of a quasi-order, the properties of which combine to yield what we call the ideal space of a WQO. We do not attempt to be comprehensive on the tentacular topic of completions of partial orders. Most of the results of Section 4.1 certainly belongs to the folklore but while most authors focus mainly on lattice theoretic or domain theoretic aspects, we concentrate on the WQO property.

In Subsection 4.1.2 we supply the definition of the so-called Cauchy ideal completion of a partial order which is studied by Erné and Palko [EP98] with a different approach. We give a characterisation of the partial orders in which the Cauchy ideals coincide with the ideals. They are the partial orders which enjoy the so-called property M – well-known in domain theory. Notably the two notions coincide in the case of a WQO, since every WQO trivially satisfies property M. Moreover for the partial orders with property M, we show that the ideal completion when equipped with the Lawson topology coincides with the Cauchy ideal completion.

Next we present the Cauchy ideal completion of a partial order P as the Priestley dual of a certain lattice of subsets of P. Following Bekkali, Pouzet, and Zhani [BPZ07] we view this as a particular case of a duality result relating the 'taking of the topological closure' with the 'algebraic generation of a lattice'. We also provide a new proof of this duality result. In particular, it turns out that the Cauchy ideal completion of a WQO is the Priestley dual of the lattice of downsets. This observation leads us in Subsection 4.1.4 to consider the profinite completion of a partial order. This is also the Nachbin order-compactification of the partial order considered with the discrete topology.

We see the coincidence of these various completions in the case of a WQO as a sign of the importance of the space of ideals of a WQO.

We then utilise the results of Chapter 3 to prove Pouzet's conjecture in Section 4.2. A slightly different proof was published by the author and R. Carroy [CP14].

We close this chapter by discussing some applications of Pouzet's conjecture in Section 4.3. We notably obtain as a corollary that an *interval order* is WQO if and only if it is BQO, as observed by Pouzet and Sauer [PS06].

Chapter 5: A Wadge Hierarchy for second countable spaces This chapter is based on an article [Peq15] published by the author in *Archive for Mathematical Logic*.

The fact that the Wadge quasi-order is well-founded on Borel subsets of ω^ω relies on the determinacy of certain infinite games and this result is actually best seen as an immediate corollary of a theorem on BQOs obtained by van Engelen, Miller, and Steel [vEMS87]. Elaborating on Chapter 2, we present in Section 5.1 a slight generalisation of this theorem (cf Theorems 5.7 and 5.9) in a way which makes it appear as an extension of the idea underlying the very definition of BQO.

From these results, we get that the quasi-order of continuous reducibility on the Borel subsets of any zero-dimensional Luzin[3] space $\mathcal{X}$ is a WQO – in fact a BQO – which satisfies the Wadge Lemma, namely for every Borel $A, B \subseteq \mathcal{X}$ either $A \leqslant_W B$ or $B \leqslant_W \mathcal{X} \setminus A$. In particular antichains have size at most 2.

The main idea of this chapter is to generalise the Wadge quasi-order to a large class of spaces while maintaining the nice properties it enjoys on the Borel subsets of the Baire space. To do this we move from reductions by continuous functions to reductions by 'continuous' relations. To begin with, we observe in Section 5.2 that total relations account perfectly for the idea of reducibility in the abstract and in fact generalise the framework of reductions as functions.

The notion of continuity for relations that fits our purpose is called *relative continuity*. It relies on the concept of *admissible representation of a topological space*. While this concept is fundamental to Type-2 Theory of Effectivity (see the textbook by Weihrauch [Wei00]), we do not expect our reader to be familiar with the simple and interesting underpinning of this approach to computable analysis. We therefore review the basic definitions and provide proofs for his convenience in Section 5.3. This Section ends with the definition of the quasi-order $\preccurlyeq_W$ of reducibility by relatively continuous relations.

We prove in Section 5.4 that the quasi-order $\preccurlyeq_W$ refines the classical hierarchies of Borel and Hausdorff–Kuratowski.

We define a general reduction game for represented spaces in Section 5.5 as a simple adaptation of the game we used in Section 5.1. This allows us to show that the quasi-order $\preccurlyeq_W$ satisfies the Wadge Lemma on Borel subsets of Borel representable spaces. Also, moving from continuous functions to relatively continuous relations we extend our version of the theorem by van Engelen, Miller, and Steel [vEMS87] from Luzin zero-dimensional spaces to all Borel representable spaces. This yields in particular that the reducibility by relatively continuous relations is well-founded – in fact BQO – on the Borel

[3] Luzin spaces are also called Borel absolute spaces.

subsets of every Borel representable space.

Finally in Section 5.6 we exemplify the difference between the continuous reducibility and the reducibility by relatively continuous relations in two major examples: the real line $\mathbb{R}$ and the Scott domain $\mathcal{P}\omega$.

2 Sequences in sets and orders

Sequences play an important rôle in both the theory of topological spaces and the theory of orders. A sequence in a set E is simply a map from the set ω of natural numbers into E. A subsequence of a sequence $f : \omega \to E$ is a restriction $f{\restriction}_X : X \to E$ of f to some infinite subset X of ω. In this chapter we focus on sequences and their generalisation, super-sequences, in the context of sets and ordered sets. This leads us from well-quasi-orders to better-quasi-orders.

2.1 Well-quasi-orders

A reflexive and transitive binary relation $\leqslant$ on a set Q is called a *quasi-order* (qo, also preorder). As it is customary, we henceforth make an abuse of terminology and refer to the pair $(Q, \leqslant)$ simply as Q when there is no danger of confusion. Moreover when it is necessary to prevent ambiguity we use a subscript and write $\leqslant_Q$ for the binary relation of the quasi-order Q.

The notion of quasi-order is certainly the most general mathematical concept of ordering. Two elements p and q of a quasi-order Q are *equivalent*, in symbols $p \equiv q$, if both $p \leqslant q$ and $q \leqslant p$ hold. It can very well happen that p is equivalent to q while p is not equal to q. This kind of situation naturally arises when one considers for example the quasi-order of embeddability among a certain class of structures. Examples of pairs of structures which mutually embed into each other while being distinct, or even non isomorphic, abound in mathematics.

Every quasi-order possess an *associated strict relation*, denoted by $<$, defined by $p < q$ if and only if $p \leqslant q$ and $q \nleqslant p$ – equivalently $p \leqslant q$ and $p \not\equiv q$. We say two elements p and q are *incomparable*, when both $p \nleqslant q$ and $q \nleqslant p$ hold, in symbols $p \mid q$.

A map $f : P \to Q$ between quasi-orders is *order-preserving* (also isotone) if whenever $p \leqslant_P p'$ holds in P we have $f(p) \leqslant_Q f(p')$ in Q. An *embedding* is a map $f : P \to Q$ such that for every p and p' in P, $p \leqslant_P p'$ if and only if $f(p) \leqslant_Q f(p')$. Notice that an embedding is not necessarily injective. An embedding $f : P \to Q$ is called an *equivalence*[1] provided it is *essentially*

[1] Viewing quasi-orders as categories in the obvious way, this notion of equivalence coincides with the one used in category theory.

surjective, i.e. for every $q \in Q$ there exists $p \in P$ with $q \equiv_Q f(p)$. We say that two quasi-orders P and Q are *equivalent* if there exists an equivalence from P to Q – by the axiom of choice this is easily seen to be an equivalence relation on the class of quasi-orders. Notice that every set X quasi-ordered by the full relation $X \times X$ is equivalent to the one point quasi-order. In contrast, by an *isomorphism* f from P to Q we mean a bijective embedding $f : P \to Q$. Of course, a set X quasi-ordered with the full relation $X \times X$ is never isomorphic to 1 except when X contains exactly one element.

In the sequel we study quasi-orders up to equivalence, namely only properties of quasi-orders which are preserved by equivalence are considered.

A quasi-order Q is called a *partial order* (po, also poset) provided the relation $\leqslant$ is antisymmetric, i.e. $p \equiv p$ implies $p = q$ – equivalent elements are equal. Notice that an embedding between partial orders is necessarily injective. Moreover if P and Q are partial orders and $f : P \to Q$ is an equivalence, then f is an isomorphism. We also note that in a partial order the associated strict order can also be defined by $p < q$ if and only if $p \leqslant q$ and $p \neq q$.

Importantly, every quasi-order Q admits up to isomorphism a unique equivalent partial order, its *equivalent partial order*, which can be obtained as the quotient of Q by the equivalence relation $p \equiv q$.

Even though most naturally occurring examples and constructions are only quasi-orders, one can always think of the equivalent partial order. The study of quasi-orders therefore really amounts to the study of partial orders.

2.1.1 Good versus bad sequences

We let $\omega = \{0, 1, 2, ...\}$ be the set of natural numbers. We use the set theoretic definitions $0 = \emptyset$ and $n = \{0, \dots, n-1\}$, so that the usual order on ω coincides with the membership relation. The equality and the usual order on ω give rise to the following distinguished types of sequences into a quasi-order.

Definitions 2.1. Let Q be a quasi-order.

(1) An *infinite antichain* is a map $f : \omega \to Q$ such that for all $m, n \in \omega$, $m \neq n$ implies $f(m) \mid f(n)$.

(2) An *infinite descending chain*, or an infinite decreasing sequence in Q, is a map $f : \omega \to Q$ such that for all $m, n \in \omega$, $m < n$ implies $f(m) > f(n)$.

(3) A *perfect sequence*, is a map $f : \omega \to Q$ such that for all $m, n \in \omega$ the relation $m \leqslant n$ implies $f(m) \leqslant f(n)$. In other words, f is perfect if it is order-preserving from $(\omega, \leqslant)$ to $(Q, \leqslant)$.

(4) A *bad sequence* is a map $f : \omega \to Q$ such that for all $m, n \in \omega$, $m < n$ implies $f(m) \nleqslant f(n)$.

(5) A *good sequence* is a map $f : \omega \to Q$ such that there exist $m, n \in \omega$ with $m < n$ and $f(m) \leqslant f(n)$. Hence a sequence is good exactly when it is not bad.

For any given set X, we denote by $[X]^2$ the set of pairs $\{x, y\}$ for distinct $x, y \in X$. By Ramsey's Theorem[2] [Ram30] whenever $[\omega]^2$ is partitioned into P_0 and P_1 there exists an infinite subset X of ω such that either $[X]^2 \subseteq P_0$, or $[X]^2 \subseteq P_1$.

Proposition 2.2. *For a quasi-order Q, the following conditions are equivalent.*

(W1) *Q has no infinite descending chain and no infinite antichain;*

(W2) *there is no bad sequence in Q;*

(W3) *every sequence in Q admits a perfect subsequence.*

Proof. (W1)→(W2): By contraposition, suppose that $f : \omega \to Q$ is a bad sequence. Partition $[\omega]^2$ into P_0 and P_1 with

$$P_0 = \Big\{ \{m, n\} \in [\omega]^2 \;\Big|\; m < n \text{ and } f(m) \ngeqslant f(n) \Big\}.$$

By Ramsey's Theorem, there exists an infinite subset X of integers with either $[X]^2 \subseteq P_0$, or $[X]^2 \subseteq P_1$. In the first case $f : X \to Q$ is an infinite antichain and in the second case $f : X \to Q$ is an infinite descending chain.

(W2)→(W1): Notice that an infinite antichain and an infinite descending chain are two examples of a bad sequence.

(W2)↔(W3): Let $f : \omega \to Q$ be any sequence in Q. We partition $[\omega]^2$ in P_0 and P_1 with

$$P_0 = \Big\{ \{m, n\} \in [\omega]^2 \;\Big|\; m < n \text{ and } f(m) \nleqslant f(n) \Big\}.$$

By Ramsey's Theorem, there exists an infinite subset X of integers such that $[X]^2 \subseteq P_0$ or $[X]^2 \subseteq P_1$. The first case yields a bad subsequence. The second case gives a perfect subsequence. □

[2]Nash-Williams' generalisation of Ramsey's Theorem is stated and proved as Theorem 2.36.

Definition 2.3. A quasi-order Q is called a *well-quasi-order* (WQO) when one of the equivalent conditions of the previous proposition is fulfilled. A quasi-order with no infinite descending chain is said to be *well-founded*.

The notion of WQO is a frequently discovered concept, for an historical account of its early development we refer the reader to the excellent article by Kruskal [Kru72].

Using Proposition 2.2 and the Ramsey's Theorem for pairs, one easily proves the following basic closure properties of the class of WQOs.

Proposition 2.4.

(i) *If $(Q, \leqslant_Q)$ is* WQO *and $P \subseteq Q$, then $(P, \leqslant_P)$ is* WQO*, where $p \leqslant_P p'$ if and only if $p, p' \in P$ and $p \leqslant_Q p'$.*

(ii) *If $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ are* WQO*, then $P \times Q$ quasi-ordered by*

$$(p, q) \leqslant_{P \times Q} (p', q') \quad \leftrightarrow \quad p \leqslant_P p' \text{ and } q \leqslant_Q q'$$

is WQO*.*

(iii) *If $(P, \leqslant_P)$ is a* partial order *and $(Q_p, \leqslant_{Q_p})$ is a quasi-order for every $p \in P$, the sum $\sum_{p \in P} Q_p$ of the Q_p along P has underlying set the disjoint union $\{(p, q) \mid p \in P \text{ and } q \in Q_p\}$ and is quasi-ordered by*

$$(p, q) \leqslant (p', q') \quad \leftrightarrow \quad \text{either } p = p' \text{ and } q \leqslant_{Q_p} q', \text{ or } p < p'.$$

If P is WQO *and each Q_p is* WQO*, then $\sum_{p \in P} Q_p$ is* WQO*.*

(iv) *If Q is* WQO *and there exists a map $g : P \to Q$ such that for all $p, p' \in P$ $g(p) \leqslant g(p') \to p \leqslant p'$, then P is* WQO*.*

(v) *If P is* WQO *and there is a surjective and monotone map $h : P \to Q$, then Q is* WQO*.*

2.1.2 Subsets and downsets

Importantly, a WQO can be characterised in terms of its subsets.

Definitions 2.5. Let Q be a quasi-order.

(1) A subset D of Q is a *downset*, or an *initial segment*, if $q \in D$ and $p \leqslant q$ implies $p \in D$. For any $S \subseteq Q$, we write $\downarrow S$ for the downset generated by S in Q, i.e. the set $\{q \in Q \mid \exists p \in S\ q \leqslant p\}$. We also write $\downarrow p$ for $\downarrow\{p\}$.

    We denote by $\mathcal{D}(Q)$ the po of downsets of Q under inclusion.

(2) We give the dual meaning to *upset*, $\uparrow S$ and $\uparrow q$ respectively.

(3) An upset U is said to to be *finitely generated*, or to admit a *finite basis*, if there exists a finite $F \subseteq U$ such that $U = \uparrow F$. We say that Q has *the finite basis property* if every upset of Q admits a finite basis.

(4) A downset $D \in \mathcal{D}(Q)$ is said to be *finitely bounded*, if there exists a finite set $F \subseteq Q$ with $D = Q \setminus \uparrow F$. We let $\mathcal{D}_{\mathrm{fb}}(Q)$ be the set of finitely bounded downsets partially ordered by inclusion.

(5) We turn the power-set of Q, denoted $\mathcal{P}(Q)$, into a qo by letting $X \leqslant Y$ if and only if $\forall p \in X\ \exists q \in Y\ p \leqslant q$, this is sometimes called the *domination quasi-order*. We let $\mathcal{P}_{<\aleph_1}(Q)$ be the the set of countable subsets of Q with the quasi-order induced from $\mathcal{P}(Q)$. Since $X \leqslant Y$ if and only if $\downarrow X \subseteq \downarrow Y$, the equivalent partial order of $\mathcal{P}(Q)$ is $\mathcal{D}(Q)$ and the quotient map is given by $X \mapsto \downarrow X$.

The notion of well-quasi-order should be thought of as a generalisation of the notion of well-ordering beyond linear orders. Recall that a po P is a linear order if for every p and q in P, either $p \leqslant q$ or $q \leqslant p$. A *well-ordering* is (traditionally, the associated strict relation $<$ of) a partial order that is both linearly ordered and well-founded.

Observe that a linearly ordered P is well-founded if and only if the initial segments of P are well-founded under inclusion. Considering for example the po $(\omega, =)$, one directly sees that a partial order P can be well-founded while the initial segments of P (here $\mathcal{P}(\omega)$) are not well-founded under inclusion. However a qo Q is WQO if and only if the initial segments of Q are well-founded under inclusion.

Proposition 2.6. *A quasi-order Q is a* WQO *if and only if one of the following equivalent conditions is fulfilled:*

(W4) *Q has the finite basis property,*

(W5) *$(\mathcal{P}(Q), \leqslant)$ is well-founded,*

(W6) *$(\mathcal{P}_{<\aleph_1}(Q), \leqslant)$ is well-founded,* (W7) $(\mathcal{D}(Q), \subseteq)$ *is well-founded,*

(W8) $(\mathcal{D}_{fb}(Q), \subseteq)$ *is well-founded.*

Proof. (W2)→(W4): By contraposition, suppose $S \in \mathcal{U}(Q)$ admits no finite basis. Since $\emptyset = \uparrow\emptyset$, $S \neq \emptyset$. By dependent choice, we can show the existence of a bad sequence $f : \omega \to Q$. Choose $f(0) \in S$ and suppose that f is defined up to some $n > 0$. Since $\uparrow\{f(0), \dots f(n)\} \subset S$ we can choose some $f(n+1)$ inside $S \setminus \uparrow\{f(0), \dots f(n)\}$.

(W4)→(W5): By contraposition, suppose that $(X_n)_{n\in\omega}$ is an infinite descending chain in $\mathcal{P}(Q)$. Then for each $n \in \omega$ we choose $q_n \in \downarrow X_n \setminus \downarrow X_{n+1}$. Then $\{q_n \mid n \in \omega\}$ has no finite basis. Indeed for all $n \in \omega$ we have $q_{n+1} \notin \uparrow\{q_i \mid i \leqslant n\}$, otherwise $q_i \leqslant q_{n+1} \in \downarrow X_{n+1} \subseteq \downarrow X_{i+1}$ for some $i \leqslant n$, a contradiction.

(W5)→(W6): Obvious.

(W6)→(W2): By contraposition, if $(q_n)_{n\in\omega}$ is a bad sequence in Q, then $P_n = \{q_k \mid n \leqslant k\}$ is an infinite descending chain in $\mathcal{P}_{<\aleph_1}(Q)$ since whenever $m < n$ we have $q_m \in P_m$ and $q_m \nleqslant q_k$ for every $k \geqslant n$.

(W5)→(W7): By contraposition, notice that an infinite descending chain in $(\mathcal{D}(Q), \subseteq)$ is also an infinite descending chain in $(\mathcal{P}(Q), \leqslant)$.

(W7)→(W8): Obvious.

(W8)→(W2): By contraposition, if $f : \omega \to Q$ is a bad sequence, then $n \mapsto D_n = Q \setminus \uparrow\{f(i) \mid i \leqslant n\}$ is an infinite descending chain in $\mathcal{D}_{\mathrm{fb}}(Q)$. □

2.1.3 Regular sequences

A monotone decreasing sequence of ordinals is, by well-foundedness, eventually constant. The limit of such a sequence exists naturally, and is simply its minimum.

In general the limit of a sequence $(\alpha_i)_{i\in\omega}$ of ordinals may not exist, however any sequence of ordinals admits a limit superior. Indeed, define the sequence $\beta_i = \sup_{j\geqslant i} \alpha_j = \bigcup_{j\geqslant i} \alpha_j$, then $(\beta_i)_{i\in\omega}$ is decreasing and hence admits a limit.

We say that $(\alpha_i)_{i\in\omega}$ is *regular* if the limit superior and the supremum $\bigcup_{i\in\omega} \alpha_i$ of $(\alpha_i)_{i\in\omega}$ coincide. This is equivalent to saying that for every $i \in \omega$ there exists $j > i$ with $\alpha_i \leqslant \alpha_j$. By induction one shows that this is in turn equivalent to saying that for all $i \in \omega$ the set $\{j \in \omega \mid i < j \text{ and } \alpha_i \leqslant \alpha_j\}$ is infinite.

Notation 2.7. For $n \in \omega$ and X an infinite subset of ω let us denote by X/n the final segment of X given by $\{k \in X \mid k > n\}$.

We generalise the definition of regular sequences of ordinals to sequences in quasi-orders as follows.

Definition 2.8. Let Q be a qo. A *regular sequence* is a map $f : \omega \to Q$ such that for all $n \in \omega$ the set $\{k \in \omega/n \mid f(n) \leqslant f(k)\}$ is infinite.

Here is a characterisation of WQO in terms of regular sequences which exhibits another property of well-orders shared by WQOs.

Proposition 2.9. *Let Q be a qo. Then Q is* WQO *if and only if one of the following equivalent conditions holds:*

(W9) *Every sequence in Q admits a regular subsequence.*

(W10) *For every sequence $f : \omega \to Q$ there exists $n \in \omega$ such that the restriction $f : \omega/n \to Q$ is regular.*

Proof. (W7)→(W10): For $f : \omega \to Q$ we let $\hat{f} : \omega \to \mathcal{D}(Q)$ be defined by $\hat{f}(n) = \downarrow\{f(k) \mid n \leqslant k < \omega\}$. Then clearly if $m < n$ then $\hat{f}(m) \supseteq \hat{f}(n)$. The partial order $\mathcal{D}(Q)$ being well-founded by (W7), there exists $n \in \omega$ such that for every $m > n$ we have $\hat{f}(n) = \hat{f}(m)$. This n is as desired. Indeed, if $k > n$ then for every $l > k$ we have $f(k) \in \hat{f}(k) = \hat{f}(l)$ and so there exists $j \geqslant l$ with $f(k) \leqslant f(l)$.

(W10)→(W9): Obvious.

(W9)→(W2): By contraposition, if $f : \omega \to Q$ is a bad sequence, then every subsequence of f is bad. Clearly a bad sequence $f : \omega \to Q$ is not regular since for every $n \in \omega$ the set $\{k \in \omega/n \mid f(n) \leqslant f(k)\}$ is empty. Hence a bad sequence admits no regular subsequence. □

2.1.4 Sequences of subsets

In this subsection we give a new characterisation of WQOs which enjoys both a topological and an order-theoretical flavour.

So far, we have considered $\mathcal{D}(Q)$ as partially ordered set for inclusion. But $\mathcal{D}(Q)$ also admits a natural topology which turns it into a compact Hausdorff 0-dimensional space. Consider Q as a discrete topological space, and form the product space 2^Q, whose underlying set is identified with $\mathcal{P}(Q)$. This product space, sometimes called *generalised Cantor space*, admits as a basis the clopen sets of the form

$$N(F,G) = \{X \subseteq Q \mid F \subseteq X \text{ and } X \cap G = \emptyset\},$$

for finite subsets F, G of Q. For $q \in Q$, we write $\langle q \rangle$ instead of $N(\{q\}, \emptyset)$ for the clopen set $\{X \subseteq Q \mid q \in X\}$. Note that $\langle q \rangle^{\complement} = N(\emptyset, \{q\})$.

Notice that $\mathcal{D}(Q)$ is an intersection of clopen sets,

$$\mathcal{D}(Q) = \bigcap_{p \leqslant q} \langle q \rangle^{\complement} \cup \langle p \rangle,$$

hence $\mathcal{D}(Q)$ is closed in 2^Q and therefore compact.

Now recall that for every sequence $(E_n)_{n\in\omega}$ of subsets of Q we have the usual relations

$$\bigcap_{n\in\omega} E_n \subseteq \bigcup_{i\in\omega} \bigcap_{j\geqslant i} E_j \subseteq \bigcap_{i\in\omega} \bigcup_{j\geqslant i} E_j \subseteq \bigcup_{n\in\omega} E_n. \tag{2.1}$$

Moreover the convergence of sequences in 2^Q can be expressed by means of a '$\liminf = \limsup$' property.

Fact 2.10. *A sequence $(E_n)_{n\in\omega}$ converges to E in 2^Q if and only if*

$$\bigcup_{i\in\omega} \bigcap_{j\geqslant i} E_j = \bigcap_{i\in\omega} \bigcup_{j\geqslant i} E_j = E.$$

Proof. Suppose that $E = \bigcup_{i\in\omega} \bigcap_{j\geqslant i} E_j = \bigcap_{i\in\omega} \bigcup_{j\geqslant i} E_j$. We show that $E_n \to E$. Let F, G be finite subsets of Q with $E \in N(F,G)$. Since $E = \bigcup_{i\in\omega} \bigcap_{j\geqslant i} E_j$ and F finite, $F \subseteq E_j$ for all sufficiently large j. Since $E = \bigcap_{i\in\omega} \bigcup_{j\geqslant i} E_j$ and G is finite, $G \cap E_j = \emptyset$ for all sufficiently large j. It follows that $E_j \in N(F,G)$ for all sufficiently large j, whence $(E_n)_n$ converges E.

Conversely, assume that E_n converges to some E in 2^Q. If q belongs to E – i.e. $E \in \langle q \rangle$ – then $q \in E_j$ for all sufficiently large j and thus $q \in \bigcup_{i\in\omega} \bigcap_{j\geqslant i} E_j$. And if $q \notin E$, i.e. $E \notin \langle q \rangle$, then $q \notin E_j$ for all sufficiently large j and thus $q \notin \bigcap_{i\in\omega} \bigcup_{j\geqslant i} E_j$. Therefore by (2.1) it follows that $E = \bigcup_{i\in\omega} \bigcap_{j\geqslant i} E_j = \bigcap_{i\in\omega} \bigcup_{j\geqslant i} E_j$. □

Observe that if $(q_n)_{n\in\omega}$ is a perfect sequence in a qo Q, then for every $q \in Q$ if $q \leqslant q_m$ for some m, then $q \leqslant q_n$ holds for all $n \geqslant m$. Therefore by (2.1) we have

$$\bigcup_{m\in\omega} \bigcap_{n\geqslant m} \downarrow q_n = \bigcup_{n\in\omega} \downarrow q_n,$$

whence $(\downarrow q_n)_{n\in\omega}$ converges to $\downarrow\{q_n \mid n \in \omega\}$ in 2^Q by Fact 2.10. On the contrary no bad sequence $(q_n)_{n\in\omega}$ converges towards $\downarrow\{q_n \mid n \in \omega\}$, since for example q_0 does not belong to $\bigcup_{i\in\omega}\bigcap_{j\geqslant i}\downarrow q_j$. We have obtained the following:

Fact 2.11. *Let Q be a qo.*

(i) *Q is* WQO *if and only if for every sequence $(q_n)_{n\in\omega}$ there exists $N \in [\omega]^\infty$ such that $(\downarrow q_n)_{n\in N}$ converges to $\downarrow\{q_n \mid n \in N\}$ in $\mathcal{D}(Q)$.*

(ii) *If Q is* WQO *and $(\downarrow q_n)_{n\in\omega}$ converges to some D in $\mathcal{D}(Q)$, then there is some $N \in [\omega]^\infty$ such that $D = \downarrow\{q_n \mid n \in N\}$.*

Actually more is true, thanks to the following ingenious observation made by Richard Rado in the body of a proof in [Rad54].

Lemma 2.12 (Rado's trick)**.** *Let Q be a* WQO *and let $(D_n)_{n\in\omega}$ be a sequence in $\mathcal{D}(Q)$. Then there exists an infinite subset N of ω such that*

$$\bigcup_{i\in N}\bigcap_{j\in N/i} D_j = \bigcup_{n\in N} D_n,$$

and so the subsequence $(D_j)_{j\in N}$ converges to $\bigcup_{n\in N} D_n$ in $\mathcal{D}(Q)$.

Proof. Towards a contradiction suppose that for all infinite $N \subseteq \omega$ we have

$$\bigcup_{i\in N}\bigcap_{j\in N/i} D_j \subset \bigcup_{n\in N} D_n. \tag{2.2}$$

We define an infinite descending chain $(E_i)_{i\in\omega}$ in $\mathcal{D}(Q)$. But to do so we recursively define a sequence $(N_k)_{k\in\omega}$ of infinite subsets of ω and a sequence $(q_k)_{k\in\omega}$ in Q such that

(a) $N_0 = \omega$ and $N_k \supseteq N_{k+1}$ for all $k \in \omega$.

(b) $q_k \in \bigcup_{j\in N_k} D_j$ and $q_k \notin \bigcup_{j\in N_{k+1}} D_j$.

Suppose we have defined $N_0, \dots, N_k$ and $q_0, \dots q_k$. By (2.2) we have

$$\bigcup_{n\in N_k} D_n \nsubseteq \bigcup_{i\in N_k}\bigcap_{j\in N_k/i} D_j,$$

so we can let $n_0 \in N_k$ and $q_k \in D_{n_0}$ be such that $q_k \notin \bigcup_{i\in N_k}\bigcap_{j\in N_k/i} D_j$. Then for all i in N_k let $j_i \in N_k/i$ be minimal such that $q_k \notin D_{j_i}$. Setting

$n_1 = j_{n_0}$ and $n_{i+1} = j_{n_i}$, we obtain an infinite set $N_{k+1} = \{n_0, n_1, n_2, ...\}$ which satisfies

$$q_k \in D_{n_0} \subseteq \bigcup_{j \in N_k} D_j \quad \text{and} \quad q_k \notin \bigcup_{j \in N_{k+1}} D_j.$$

Now we define $E_k = \bigcup_{j \in N_k} D_j$. The sequence $(E_k)_{k\in\omega}$ is an infinite descending chain in $\mathcal{D}(Q)$, contradicting the fact that Q is WQO.

For the second statement, observe that if N is an infinite subset of ω satisfying the statement of the lemma, then by (2.1) we have

$$\bigcup_{i\in N} \bigcap_{j\in N/i} D_j = \bigcap_{i\in N} \bigcup_{j\in N/i} D_j = \bigcup_{n\in N} D_n,$$

and so by Fact 2.10 we get that $(D_j)_{j\in N}$ converges to $\bigcup_{n\in N} D_n$ in $\mathcal{D}(Q)$. □

Hence if Q is WQO, then every sequence in $\mathcal{D}(Q)$ admits a subsequence which converges to its union. Of course the converse also holds.

Lemma 2.13. *If $(D_n)_{n\in\omega}$ is an infinite descending chain in $\mathcal{D}(Q)$, then there is no infinite subset N of ω such that $\bigcup_{n\in N} D_n = \bigcup_{i\in N} \bigcap_{j\in N/i} D_j$.*

Proof. Since any subsequence of an infinite descending chain is again an infinite descending chain, it is enough to show that if $(D_n)_{n\in\omega}$ is an infinite descending chain in $\mathcal{D}(Q)$ then $\bigcup_{n\in\omega} D_n \nsubseteq \bigcup_{i\in\omega} \bigcap_{j\in\omega/i} D_j$. Pick any $q \in D_0 \setminus D_1$. Then since $D_j \subseteq D_1$ for all $j \geqslant 1$ and D_1 is a down set, we get $q \notin D_j$ for all $j \geqslant 1$. It follows that $q \notin \bigcup_{i\in\omega} \bigcap_{j\in\omega/i} D_j$. □

This leads to our last characterisation of WQO:

Proposition 2.14. *Let Q be a qo. Then Q is* WQO *if and only if*

(W11) *Every sequence $(D_n)_{n\in\omega}$ in $\mathcal{D}(Q)$ admits a subsequence $(D_n)_{n\in N}$ which converges to $\bigcup_{n\in N} D_n$.*

2.2 Better-quasi-orders

2.2.1 Towards better

As we have seen in Proposition 2.6 a quasi-order is WQO if and only if $\mathcal{P}(Q)$ is well-founded if and only if $\mathcal{D}(Q)$ is well founded. The first example of a WQO whose powerset contains an infinite antichain was identified by Richard Rado. This WQO is the starting point of the journey towards the stronger notion of a better-quasi-order and it will stand as a crucial example in this thesis.

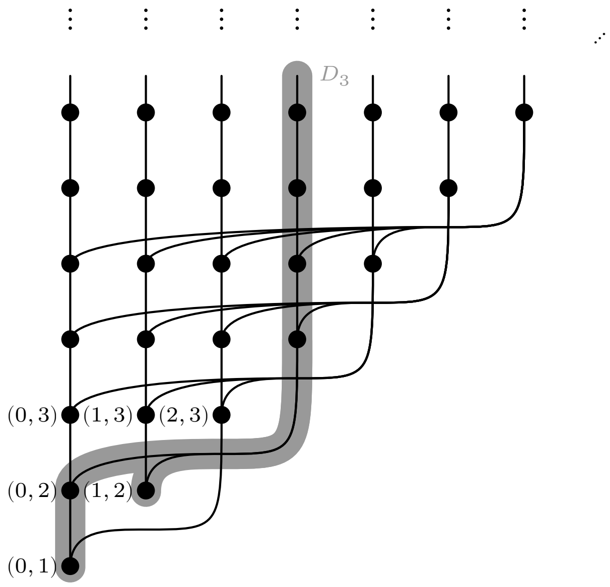


Figure 2.1: Rado's poset $\mathfrak{R}$.

Example 2.15 ([Rad54]). Rado's partial order $\mathfrak{R}$ is the set $[\omega]^2$, of pairs of natural numbers, partially ordered by (cf. Figure 2.1):

$$\{m, n\} \leqslant \{m', n'\} \quad \leftrightarrow \quad \begin{cases} m = m' \text{ and } n \leqslant n', \text{ or} \\ n < m'. \end{cases}$$

where by convention a pair $\{m, n\}$ of natural numbers is always assumed to be written in increasing order ($m < n$).

The po $\mathfrak{R}$ is WQO. To see this, consider any map $f : \omega \to [\omega]^2$ and let $f(n) = \{f_0(n), f_1(n)\}$ for all $n \in \omega$. Now if f_0 is unbounded, then there exists $n > 0$ with $f_1(0) < f_0(n)$ and so $f(0) \leqslant f(n)$ in $\mathfrak{R}$ by the second clause. If f_0 is bounded, let us assume by going to a subsequence if necessary, that $f_1 : \omega \to \omega$ is perfect. Then there exist m and n with $m < n$ and $f_0(m) = f_0(n)$ and we have $f_1(m) \leqslant f_1(n)$, so $f(m) \leqslant f(n)$ in $\mathfrak{R}$ by the first clause. In both case we find that f is good, so $\mathfrak{R}$ is WQO.

However the map $n \mapsto D_n = \downarrow\{\{n, l\} \mid n < l\}$ is a bad sequence (in fact an infinite antichain) inside $\mathcal{D}(\mathfrak{R})$. Indeed whenever $m < n$ we have $\{m, n\} \in D_m$ while $\{m, n\} \notin D_n$, and so $D_m \nsubseteq D_n$.

One natural question is now: What witnesses in a given quasi-order Q the fact that $\mathcal{P}(Q)$ is not WQO? It cannot always be a bad sequence, that is what the existence of Rado's poset tells us. But then what is it?

To see this suppose that $(P_n)_{n\in\omega}$ is a bad sequence in $\mathcal{P}(Q)$. Fix some $m \in \omega$. Then whenever $m < n$ we have $P_m \not\subseteq \downarrow P_n$ and we can choose a witness $q \in P_m \setminus \downarrow P_n$. But of course in general there is no single $q \in P_m$ that witnesses $P_m \not\subseteq \downarrow P_n$ for all $n > m$. So we are forced to pick a sequence $f_m : \omega/m \to Q$, $n \mapsto q_m^n$ of witnesses:

$$q_m^n \in P_m \quad \text{and} \quad q_m^n \notin \downarrow P_n, \qquad n \in \omega/m.$$

Bringing together all the sequences $f_0, f_1, \ldots$, we obtain a *sequence of sequences*, naturally indexed by the set $[\omega]^2$ of pairs of natural numbers,

$$\begin{aligned} f : [\omega]^2 &\longrightarrow Q \\ \{m, n\} &\longmapsto f_m(n) = q_m^n. \end{aligned}$$

By our choices this sequence of sequences satisfies the following condition:

$$\forall m, n, l \in \omega \quad m < n < l \to q_m^n \nleqslant q_n^l.$$

Indeed, suppose towards a contradiction that for $m < n < l$ we have $q_m^n \leqslant q_n^l$. Since $q_n^l \in P_n$ we would have $q_m^n \in \downarrow P_n$, but we chose q_m^n such that $q_m^n \notin \downarrow P_n$.

Let us say that a sequence of sequences $f : [\omega]^2 \to Q$ is *bad* if for every $m, n, l \in \omega$, $m < n < l$ implies $f(\{m, n\}) \nleqslant f(\{n, l\})$. We have come to the following.

Proposition 2.16. *Let Q be a qo. Then $\mathcal{P}(Q)$ is* WQO *if and only if there is no bad sequence of sequences into Q.*

Proof. As we have seen in the preceding discussion, if $\mathcal{P}(Q)$ is not WQO then from a bad sequence in $\mathcal{P}(Q)$ we can make choices in order to define a bad sequences of sequences in Q.

Conversely, if $f : [\omega]^2 \to Q$ is a bad sequence of sequences, then for each $m \in \omega$ we can consider the set $P_m = \{f(\{m, n\}) \mid n \in \omega/m\}$ consisting in the image of the m^{th} sequence. Then the sequence $m \mapsto P_m$ in $\mathcal{P}(Q)$ is a bad sequence. Indeed every time $m < n$ we have $f(\{m, n\}) \in P_m$ while $f(\{m, n\}) \notin \downarrow P_n$, since otherwise there would exist $l > n$ with $f(\{m, n\}) \leqslant f(\{n, l\})$, a contradiction with the fact that f is a bad sequence of sequences. □

One should notice that from the previous proof we actually get that $\mathcal{P}(Q)$ is WQO if and only if $\mathcal{P}_{<\aleph_1}(Q)$ is WQO. Notice also that in the case of Rado's partial order $\mathfrak{R}$ the fact that $\mathcal{P}(\mathfrak{R})$ is not WQO is witnessed by the bad sequence $f : [\omega]^2 \to \mathfrak{R}, \{m, n\} \mapsto \{m, n\}$ which is simply the identity on the underlying sets, since every time $m < n < l$ then $\{m, n\} \nleqslant \{n, l\}$ in $\mathfrak{R}$. In fact, Rado's partial order is in a sense universal as established by Richard Laver [Lav76]:

Theorem 2.17. *If Q is* WQO *but $\mathcal{P}(Q)$ is not* WQO*, then $\mathfrak{R}$ embeds into Q.*

Proof. Let $f : [\omega]^2 \to Q$ be a bad sequence of sequences. Partitioning the triples $\{i,j,k\}$, $i<j<k$, into two sets depending on whether or not $f(\{i,j\}) \leqslant f(\{i,k\})$, we get by Ramsey's Theorem an infinite set $N \subseteq \omega$ whose triples are all contained into one of the classes. If for every $\{i,j,k\} \subseteq N$ we have $f(\{i,j\}) \nleqslant f(\{i,k\})$ then for any $i \in N$ the sequence $f(\{i,j\})_{j\in N/i}$ is a bad sequence in Q. Since Q is WQO, the other possibility must hold.

Then partition the quadruples $\{i,j,k,l\}$ in N into two sets according to whether or not $f(\{i,j\}) \leqslant f(\{k,l\})$. Again there exists an infinite subset M of N whose quadruples are all contained into one of the classes. If all quadruples $\{i,j,k,l\}$ in M satisfy $f(\{i,j\}) \nleqslant f(\{k,l\})$, then for any sequence $(\{i_k,j_k\})_{k\in\omega}$ of pairs in M with $j_k < i_{k+1}$ the sequence $f(\{i_k,j_k\})_{k\in\omega}$ is bad in Q. Since Q is WQO, it must be the other possibility that holds.

Let $X = M \setminus \{\min M\}$, then $\{f(\{i,j\}) \mid \{i,j\} \in [X]^2\}$ is isomorphic to $\mathfrak{R}$. By the properties of M, we have $\{i,j\} \leqslant \{k,l\}$ in $\mathfrak{R}$ implies $f(\{i,j\}) \leqslant f(\{k,l\})$. We show that $f(\{i,j\}) \leqslant f(\{k,l\})$ implies $\{i,j\} \leqslant \{k,l\}$ in $\mathfrak{R}$. Suppose $\{i,j\} \nleqslant \{k,l\}$ in $\mathfrak{R}$, namely $k \leqslant j$ and either $i \neq k$, or $l < j$. If $l < j$ and $f(\{i,j\}) \leqslant f(\{k,l\})$ then for any $n \in X/j$ we have $f(\{k,l\}) \leqslant f(\{j,n\})$ and thus $f(\{i,j\}) \leqslant f(\{j,n\})$ a contradiction since f is bad. Suppose now that $k \leqslant j$ and $i \neq k$. If $i < k$ and $f(\{i,j\}) \leqslant f(\{k,l\})$, then $f(\{i,k\}) \leqslant f(\{i,j\}) \leqslant f(\{k,l\})$, a contradiction. Finally if $k < i$ and $f(\{i,j\}) \leqslant f(\{k,l\})$ then for $m = \min M$ we have $f(\{m,k\}) \leqslant f(\{i,j\}) \leqslant f(\{k,l\})$, again a contradiction. □

From a heuristic viewpoint, a *better-quasi-order* is a well quasi-order Q such that $\mathcal{P}(Q)$ is WQO, $\mathcal{P}(\mathcal{P}(Q))$ is WQO, $\mathcal{P}(\mathcal{P}(\mathcal{P}(Q)))$ is WQO, so on and so forth, into the transfinite. This idea will be made precise in Subsection 2.2.4, but we can already see that it cannot serve as a convenient definition[3]. As the above discussion suggests, a better-quasi-order is going to be a qo Q, with no bad sequence, with no bad sequence of sequences, no bad sequence of sequences of sequences, so on and so forth, into the transfinite. To do so we need a convenient notion of 'index set' for a sequence of sequences of … of sequences, in short a *super-sequence*. We now turn to the study of this fundamental notion defined by Nash-Williams.

[3]The reader who remains unconvinced can try to prove that the partial order $(2,=)$ satisfies this property.

2.2.2 Super-sequences

Let us first introduce some useful notations. Given an infinite subset X of ω and a natural number k, we denote by $[X]^k$ the set of subsets of X of cardinality k, and by $[X]^{<\infty}$ the set $\bigcup_{k\in\omega}[X]^k$ of finite subsets of X. When we write an element $s \in [X]^k$ as $\{n_0, \dots, n_{k-1}\}$ we always assume it is written in increasing order $n_0 < n_1 < \dots < n_{k-1}$ for the usual order on ω. The cardinality of $s \in [\omega]^{<\infty}$ is denoted by $|s|$. We write $[X]^\infty$ for the set of infinite subsets of X.

For any $X \in [\omega]^\infty$ and any $s \in [\omega]^{<\infty}$, we let $X/s = \{k \in X \mid \max s < k\}$ and we write X/n for $X/\{n\}$, as we have already done.

Index sets for super-sequences

Intuitively super-sequences are sequences of sequences … of sequences. In order to deal properly with this idea we need a convenient notion of index sets. Those will be families of finite sets of natural numbers called *fronts*. They were defined by Nash-Williams [Nas65]. As the presence of an ellipsis in the expression ‘sequences of sequences of … of sequences’ suggests, the notion of front admits an inductive definition. To formulate such a definition it is useful to identify the degenerate case of a super-sequence, the level zero of the notion of sequence of … of sequences, namely a function $f : 1 \to E$ which singles out a point of a set E. The index set for these degenerate sequences is the family $\{\emptyset\}$ called the *trivial front*. New fronts are then built up from old ones using the following operation.

Definition 2.18. If $X \in [\omega]^\infty$ and $F(n) \subseteq [X/n]^{<\infty}$ for every $n \in X$, we let

$$\operatorname*{seq}_{n\in X} F(n) = \big\{\{n\} \cup s \mid n \in X \text{ and } s \in F(n)\big\}.$$

Definition 2.19 (Front, inductive definition)**.** A family F for $X \in [\omega]^\infty$ is a *front on X* if it belongs to the smallest set of families of finite sets of natural numbers such that

(1) for all $X \in [\omega]^\infty$, the family $\{\emptyset\}$ is a front on X,

(2) if $X \in [\omega]^\infty$ and if $F(n)$ is a front on X/n for all $n \in X$, then

$$F = \operatorname*{seq}_{n\in X} F(n)$$

is a front on X.

Remark 2.20. In the literature, fronts are sometimes called *blocks* or *thin blocks*. Since in Section 3.2 and Chapter 3 we have another use for the term *block*, we follow the terminology of Todorčević [Tod10].

Examples 2.21. We have already seen example of fronts. Indeed for every $X \in [\omega]^\infty$ and every $n \in \omega$ the family $[X]^n$ is a front on X, where $[X]^0 = \{\emptyset\}$ is the trivial front. For a new example, consider for every $n \in \omega$ the front $[\omega/n]^n$ and build

$$\mathcal{S} = \operatorname*{seq}_{n\in\omega} [\omega/n]^n = \{s \in [\omega]^{<\infty} \mid 1 + \min s = |s|\}.$$

The front $\mathcal{S}$ is traditionally called the *Schreier barrier*.

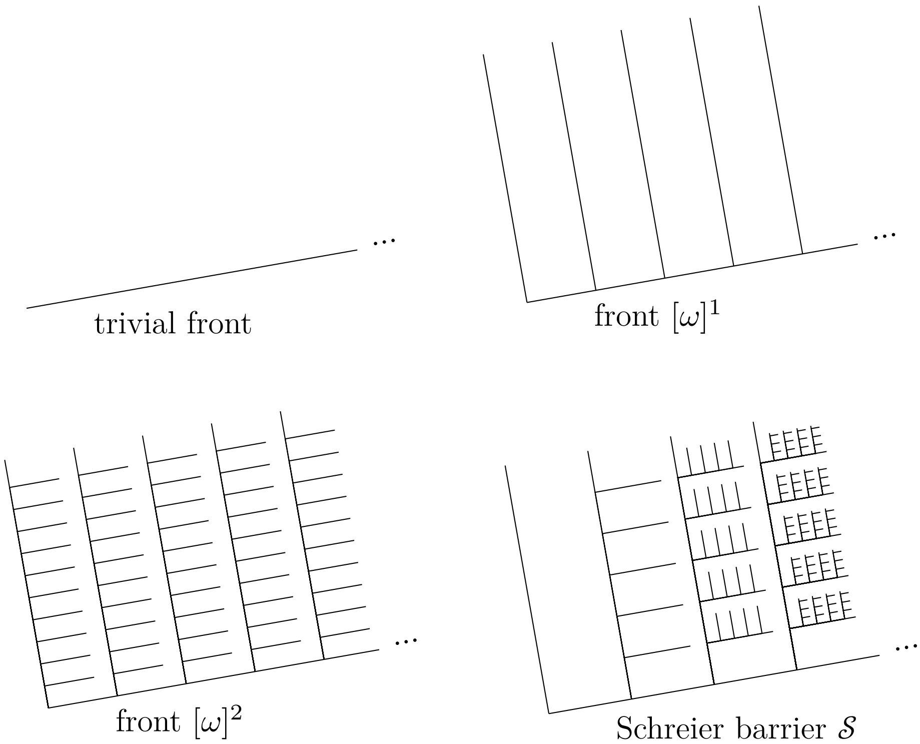


Figure 2.2: Pictures of fronts

We defined fronts to make the following:

Definition 2.22. A *super-sequence* in a set E is a map $f : F \to E$ from a front into E.

Notice that if F is a non trivial front on X, we can recover the unique sequence $F(n)$, $n \in X$, of fronts from which it is constructed.

Definition 2.23. For any family $F \subseteq [\omega]^{<\infty}$ and $n \in \omega$ we define the *ray* of F at n to be the family

$$F_n = \{s \in [\omega/n]^{<\infty} \mid \{n\} \cup s \in F\}.$$

Then every non trivial front F on X is built up from its rays F_n, $n \in X$, in the sense that:

$$F = \operatorname*{seq}_{n \in X} F_n.$$

Notice that, according to our definition, the trivial front $\{\emptyset\}$ is a front on X for every $X \in [\omega]^{\infty}$. Except for this degenerate example, if a family $F \subseteq [X]^{<\infty}$ is a front on X, then necessarily X is equal to $\bigcup F$, the set-theoretic union of the family F. For this reason we will sometimes say that F is a front, without reference to any infinite subset X of ω. Moreover when F is not trivial, we refer to the unique X for which F is a front on X, namely $\bigcup F$, as the *base* of F.

Importantly, the notion of a front also admits an explicit definition to which we now turn. It makes essential use of the following binary relation.

Definition 2.24. For subsets u, v of ω, we write $u \sqsubseteq v$ when u is an *initial segment* of v, i.e. when $u = v$ or when there exists $n \in v$ such that $u = \{k \in v \mid k < n\}$. As usual, we write $u \sqsubset v$ for $u \sqsubseteq v$ and $u \neq v$.

Definition 2.25 (Front, explicit definition)**.** A family $F \subseteq [\omega]^{<\infty}$ is a *front on* $X \in [\omega]^{\infty}$ if

(1) either $F = \{\emptyset\}$, or $\bigcup F = X$,

(2) for all $s, t \in F$ $s \sqsubseteq t$ implies $s = t$,

(3) (Density) for all $X' \in [X]^{\infty}$ there is an $s \in F$ such that $s \sqsubset X'$.

Merely for the purpose of showing that our two definitions coincide, and only until this is achieved, let us refer to a front according to the explicit definition as a front$^{\mathrm{e}}$. Notice that the family $\{\emptyset\}$ is a front$^{\mathrm{e}}$, the trivial front$^{\mathrm{e}}$. Notice also that if F is a non trivial front$^{\mathrm{e}}$ then necessarily $\emptyset \notin F$.

Our first step towards proving the equivalence of our two definitions of fronts is the following easy observation.

Lemma 2.26. *Let F be a non trivial front^{e} on $X \in [\omega]^{\infty}$. Then for every $n \in X$, the ray F_n is a front^{e} on X/n. Moreover $F = \operatorname{seq}_{n \in X} F_n$.*

Proof. Let $n \in X$. For every $Y \in [X/n]^\infty$ there exists $s \in F$ with $s \sqsubset \{n\} \cup Y$. Since F is non trivial, $s \neq \emptyset$ and so $n \in s$. Therefore $s' = s \setminus \{n\} \in F_n$ with $s' \sqsubset Y$, and F_n satisfies (3). Now if F_n is not trivial and $k \in X/n$, there is $s \in F_n$ with $s \sqsubset \{k\} \cup X/k$ and necessarily $k \in s \subseteq \bigcup F_n$. Hence $\bigcup F_n = X/n$, so condition (1) is met. To see (2), let $s, t \in F_n$ with $s \sqsubseteq t$. Then for $s' = \{n\} \cup s$ and $t' = \{n\} \cup t$ we have $s', t' \in F$ and $s' \sqsubseteq t'$, so $s' = t'$ and $s = t$, as desired. The last statement is obvious. □

Our next step consists in assigning a rank to every front[e]. To do so, we first recall some classical notions about sequences and trees.

Notations 2.27. For a non empty set A, we write A^n for the set of sequences $s : n \to A$. Let $A^{<\omega}$ be the set $\bigcup_{n \in \omega} A^n$ of finite sequences in A. We write A^ω for the set of infinite sequences $x : \omega \to A$ in A. Let $u \in A^{<\omega}$, $x \in A^{<\omega} \cup A^\omega$.

(1) $|x| \in \omega + 1$ denotes the length of x.

(2) For $n \leqslant |x|$, $x{\restriction}_n$ is the initial segment, or prefix, of x of length n.

(3) We write $u \sqsubseteq x$ if there exists $n \leqslant |x|$ with $u = x{\restriction}_n$. We write $u \sqsubset x$ if $u \sqsubseteq x$ and $u \neq x$.

(4) We write $u \frown x$ for the concatenation operation.

Identifying any finite subset of ω with its increasing enumeration with respect to the usual order on ω, we view any front[e] as a subset of $\omega^{<\omega}$. Notice that under this identification, our previous definition of $\sqsubseteq$ for subsets of ω coincides with the one for sequences.

Definitions 2.28. (1) A *tree* T on a set A is a subset of $A^{<\omega}$ that is closed under prefixes, i.e. $u \sqsubseteq v$ and $v \in T$ implies $u \in T$.

(2) A tree T on A is called *well-founded* if T has no infinite branch, i.e. if there is no infinite sequence $x \in A^\omega$ such that $x{\restriction}_n \in T$ holds for all $n \in \omega$. In other words, a tree T is well-founded if $(T, \sqsupseteq)$ is a well-founded partial order.

(3) When T is a non-empty well-founded tree we can define a strictly decreasing function ρ_T from T to the ordinals by transfinite recursion on the well-founded relation $\sqsupset$:

$$\rho_T(t) = \sup\{\rho_T(s) + 1 \mid t \sqsubset s \in T\} \quad \text{for all } t \in T.$$

It is easily shown to be equivalent to

$$\rho_T(t) = \sup\{\rho_T(t \frown (a)) + 1 \mid a \in A \text{ and } t \frown (a) \in T\} \quad \text{for all } t \in T.$$

The *rank* of the non-empty well-founded tree T is the ordinal $\rho_T(\emptyset)$.

For any front$^{\mathrm{e}}$ F, we let $T(F)$ be the smallest tree on ω containing F, i.e.

$$T(F) = \{s \in \omega^{<\omega} \mid \exists t \in F\ s \sqsubseteq t\}.$$

The following is a direct consequence of the explicit definition of a front.

Lemma 2.29. *For every front^{e} F, the tree $T(F)$ is well-founded.*

Proof. If x is an infinite branch of $T(F)$, then x enumerates an infinite subset X of $\bigcup F$ such that for every $u \sqsubset X$ there exists $t \in F$ with $u \sqsubseteq t$. Since F is a front$^{\mathrm{e}}$ there exists a (unique) $s \in F$ with $s \sqsubset X$. But for $n = \min X/s$, $u = s \cup \{n\}$ and there is $t \in F$ with $u \sqsubseteq t$. But then $F \ni s \sqsubset u \sqsubseteq t \in F$ contradicting the explicit definition of a front. □

Definition 2.30. Let F be a front$^{\mathrm{e}}$. The *rank* of F, denoted by $\operatorname{rk} F$, is the rank of the tree $T(F)$.

Example 2.31. Notice that the family $\{\emptyset\}$ is the only front$^{\mathrm{e}}$ of null rank, and for all positive integer n, the front $[\omega]^n$ has rank n. Moreover the Schreier barrier $\mathcal{S}$ has rank ω.

We now observe that the rank of F is closely related to the rank of its rays F_n, $n \in X$. Let F be a non trivial front$^{\mathrm{e}}$ on $X \in [\omega]^\infty$ and recall that by Lemma 2.26, the ray F_n is a front$^{\mathrm{e}}$ on X/n for every $n \in X$. Now notice that the tree $T(F_n)$ of the front$^{\mathrm{e}}$ F_n is naturally isomorphic to the subset

$$\{s \in T(F) \mid \{n\} \sqsubseteq s\}$$

of $T(F)$. The rank of the front$^{\mathrm{e}}$ F is therefore related to the ranks of its rays through the following formula:

$$\operatorname{rk} F = \sup\{\operatorname{rk}(F_n) + 1 \mid n \in X\}.$$

In particular, $\operatorname{rk} F_n < \operatorname{rk} F$ for all $n \in X$.

This simple remark allows one to prove results on front$^{\mathrm{e}}$s by induction on the rank by applying the induction hypothesis to the rays, as it was first done by Pudlák and Rödl [PR82]. It also allows us to prove that the two definitions of a front that we gave actually coincide.

Lemma 2.32. *The explicit definition and the inductive definition of a front coincide.*

Proof. *Inductive → Explicit:* The family $\{\emptyset\}$ is the trivial front^{e}. Now let $X \in [\omega]^\infty$ and suppose that F_n is a front^{e} on X/n for all $n \in X$. We need to see that $F = \mathrm{seq}_{n\in X} F_n$ is a front^{e} on X. Clearly $\bigcup F = X$. If $s, t \in F$ and $s \sqsubseteq t$, then $\min s = \min t$ and so for $s' = s \setminus \{\min s\}$ and $t' = t \setminus \{\min t\}$ we have $s', t' \in F_n$ and $s' \sqsubseteq t'$, hence $s' = t'$ and $s = t$. Finally, if $Y \in [X]^\infty$ with $n = \min Y$, then there exists $s' \in F_n$ with $s' \sqsubset Y \setminus \{n\}$ and so $s = \{n\} \cup s' \in F$ and $s \sqsubset Y$. So F is a front^{e}, as desired.

Explicit → Inductive: We show that every front^{e} F satisfies the inductive definition of a front by induction on the rank of F. If $\mathrm{rk}\, F = 0$, then $F = \{\emptyset\}$ is a front according to the inductive definition. Now suppose F is a front according to the explicit definition with $\mathrm{rk}\, F > 0$. In particular $\bigcup F = X$ for some $X \in [\omega]^\infty$. By Lemma 2.26, F_n is a front^{e} for every $n \in X$, and as $\mathrm{rk}\, F_n < \mathrm{rk}\, F$ for every $n \in X$, it follows that F_n is a front according to the inductive definition, by induction hypothesis. Finally as $F = \mathrm{seq}_{n\in X} F_n$, we get that F is a front according to the inductive definition. □

Finally notice that the rank of a front naturally arise from the inductive definition. Let $\mathfrak{F}_0$ be the set containing only the trivial front. Then for any countable ordinal α, let $F \in \mathfrak{F}_\alpha$ if $F \in \bigcup_{\beta<\alpha} \mathfrak{F}_\beta$ or $F = \mathrm{seq}_{n\in X} F_n$ where $X \in [\omega]^\infty$ and each F_n is a front on X/n which belongs to some $\mathfrak{F}_{\beta_n}$ for some $\beta_n < \alpha$. Then clearly the set of all fronts is equal $\bigcup_{\alpha<\omega_1} \mathfrak{F}_\alpha$. Now it should be clear that for every front F the smallest $\alpha < \omega_1$ for which $F \in \mathfrak{F}_\alpha$ is $\mathrm{rk}\, F$, the rank of F.

Sub-front and sub-super-sequences

When dealing with arbitrary super-sequences we will be particularly interested in extracting sub-super-sequences which enjoy further properties.

Definition 2.33. A sub-super-sequence of a super-sequence $f : F \to E$ is a restriction $f\restriction_G : G \to E$ to some front G included in F.

The following important operation allows us to understand the *sub-fronts* of a given front, i.e. sub-families of a front which are themselves fronts. For a family $F \subseteq \mathcal{P}(\omega)$ and some $X \in [\omega]^\infty$, we define the sub-family

$$F|X := \{s \in F \mid s \subseteq X\}.$$

Proposition 2.34. *Let F be a front on X. Then a family $F' \subseteq F$ is a front if and only if there exists $Y \in [X]^\infty$ such that $F|Y = F'$.*

Proof. The claim is obvious if F is trivial so suppose F is non-trivial.

→ Let $F' \subseteq F$ be a front on Y. Since F' is not trivial either, $Y = \bigcup F' \subseteq \bigcup F = X$. Now if $s \in F'$ then clearly $s \in F|Y$. Conversely if $s \in F|Y$ then there exists a unique $t \in F'$ with $t \sqsubset s \cup Y/s$ and so either $s \sqsubseteq t$ or $t \sqsubseteq s$. Since F is a front and $s, t \in F$, necessarily $s = t$ and so $s \in F'$. Therefore $F' = F|Y$.

← If $Y \in [X]^\infty$ then the family $F|Y$ is a front on Y. Clearly $F|Y$ satisfies (2). If $Z \in [Y]^\infty$ then since $Y \subseteq X$, then $Z \in [X]^\infty$ and so there exists $s \in F$ with $s \sqsubset Z$. But then $s \subseteq Z \subseteq Y$, so in fact $s \in F|Y$ and therefore $F|Y$ satisfies (3). For (1), notice that $\bigcup F|Y \subseteq Y$ by definition and that if $n \in Y$, then as we have already seen there exists $s \in F|Y$ with $s \sqsubset \{n\} \cup Y/n$, so $n \in s$ and $n \in \bigcup F|Y$. □

Observe that the operation of restriction commutes with the taking of rays.

Fact 2.35. *Let $F \subseteq \mathcal{P}(\omega)$ and $X \in [\omega]^\infty$. For every $n \in X$ we have*

$$F_n|X = (F|X)_n.$$

Notice also the following simple important fact. If F' is a sub-front of a front F, then the tree $T(F')$ is included in the tree $T(F)$ and so $\operatorname{rk} F' \leqslant \operatorname{rk} F$.

Throughout this thesis we extensively make use of the following fundamental theorem by Nash-Williams: Any time we partition a front into finitely many pieces, at least one of the pieces must contain a front.

Theorem 2.36 (Nash-Williams)**.** *Let F be a front. For any subset S of F there exists a front $F' \subseteq F$ such that either $F' \subseteq S$ or $F' \cap S = \emptyset$.*

We now prove this theorem to give a simple example of a proof by induction on the rank of a front, a technique we make use of on several occasions in Chapter 3.

Proof. The claim is obvious for the trivial front whose only subsets are the empty set and the whole trivial front. So suppose that the claim holds for every front of rank smaller than α. Let F be a front on X with $\operatorname{rk} F = \alpha$ and $S \subseteq F$. For every $n \in X$ let S_n be the subset of the ray F_n given by $S_n = \{s \in F_n \mid \{n\} \cup s \in S\}$.

Set $X_{-1} = X$ and $n_0 = \min X_{-1}$. Since $\operatorname{rk} F_{n_0} < \alpha$ there exists by induction hypothesis some $X_0 \in [X_{-1}/n_0]^\infty$ such that

$$\text{either } F_{n_0}|X_0 \subseteq S_{n_0}, \quad \text{or } F_{n_0}|X_0 \cap S_{n_0} = \emptyset.$$

Set $n_1 = \min X_0$. Now applying the induction hypothesis to $F_{n_1}|(X_0/n_0)$ and S_{n_1} we get an $X_1 \in [X_0/n_0]^\infty$ such that either $F_{n_1}|X_1 \subseteq S_{n_1}$, or $F_{n_1}|X_1 \cap S_{n_1} = \emptyset$. Continuing in this fashion, we obtain a sequence X_k together with $n_k = \min X_{k-1}$ such that for all k we have $X_k \in [X_{k-1}/n_k]^\infty$ and

$$\text{either } F_{n_k}|X_k \subseteq S_{n_k}, \quad \text{or } F_{n_k}|X_k \cap S_{n_k} = \emptyset.$$

Now there exists $Y \in [\omega]^\infty$ such that either $F_{n_k}|X_k \subseteq S_{n_k}$ for all $k \in Y$, or $F_{n_k}|X_k \cap S_{n_k} = \emptyset$ for all $k \in Y$. Let $X = \{n_k \mid k \in Y\}$ then $F|X$ is as desired. Indeed for all $s \in F|X$ we have $\min s = n_k$ for some $k \in Y$ and $s \setminus \{n_k\} \in F_{n_k}|X_k$. Hence by the choice of Y, either $s \setminus \{\min s\} \in S_{\min s}$ for all $s \in F|X$, or $s \setminus \{\min s\} \notin S_{\min s}$ for all $s \in F|X$. Therefore either $F|X \subseteq S$ or $F|X \cap S = \emptyset$. □

2.2.3 Multi-sequences

Another approach to super-sequences initiated by Simpson [Sim85] has proved very useful in the theory of better-quasi-orders. We now describe this approach and relate it to super-sequences.

Let E be any set, and $f : F \to E$ be a super-sequence with F a front on X. By the explicit definition of front for every $Y \in [X]^\infty$ there exists a unique $s \in F$ with $s \sqsubset Y$. We can therefore define a map $f^\uparrow : [X]^\infty \to E$ defined by $f^\uparrow(Y) = f(s)$ where s is the unique member of F with $s \sqsubset Y$.

Definition 2.37. A *multi-sequence* into some set E is a map $h : [X]^\infty \to E$ for some $X \in [\omega]^\infty$. A *sub-multi-sequence* of $h : [X]^\infty \to E$ is a restriction of h to $[Y]^\infty$ for some $Y \in [X]^\infty$.

For every $X \in [\omega]^\infty$ we endow $[X]^\infty$ with the topology induced by the Cantor space, viewing subsets as their characteristic functions. As a topological space $[X]^\infty$ is homeomorphic to the Baire space ω^ω. This homeomorphism is conveniently realised via the embedding of $[X]^\infty$ into ω^ω which maps each $Y \in [X]^\infty$ to its injective and increasing enumeration $e_Y : \omega \to Y$. We henceforth identify the space $[X]^\infty$ with the closed subset of ω^ω of injective and increasing sequences in X. From this point of view we have a countable basis of clopen sets for $[X]^\infty$ consisting in sets of the form

$$M_s = N_s \cap [X]^\infty = \{Y \in [X]^\infty \mid s \sqsubset Y\}, \quad \text{for } s \in [X]^{<\infty}.$$

Definition 2.38. A multi-sequence $h : [X]^\infty \to E$ is *locally constant* if for all $Y \in [X]^\infty$ there exists $s \in [X]^{<\infty}$ such that $Y \in M_s$ and h is constant on M_s, i.e. for every $Y \in [X]^\infty$ there exists $s \sqsubset Y$ such that for every $Z \in [X]^\infty$, $s \sqsubset Z$ implies $h(Z) = h(Y)$.

Clearly for every super-sequence $f : F \to E$ where F is a front on X the map $f^\uparrow : [X]^\infty \to E$ is locally constant.

Conversely for any locally constant multi-sequence $h : [X]^\infty \to E$, let

$$S^h = \{s \in [X]^{<\infty} \mid h \text{ is constant on } M_s\}.$$

Lemma 2.39. *The set F^h of $\sqsubseteq$-minimal elements of S^h is a front on X.*

Proof. By $\sqsubseteq$-minimality if $s, t \in F^h$ and $s \sqsubseteq t$, then $s = t$. For every $Y \in [X]^\infty$, since h is locally constant there exists $s \sqsubset Y$ such that h is constant on M_s. Hence there exists $t \in F^h$ with $t \sqsubseteq s$, and so $t \sqsubset Y$ too. To see that either F^h is trivial or $\bigcup F^h = X$, notice that h is constant if and only if F^h is the trivial front if and only if $\emptyset \in F_h$. So if F^h is not trivial, then for every $n \in X$ there exists $s \in F^h$ with $s \sqsubset \{n\} \cup X/n$ and since $s \neq \emptyset$, we get $n \in s$ and $n \in \bigcup F^h$. □

We can therefore associate to every locally constant multi-sequence $h : [X]^\infty \to E$ a super-sequence $h^\downarrow : F^h \to E$ by letting, in the obvious way, $h^\downarrow(s)$ be equal to the unique value taken by h on M_s for every $s \in F^h$.

Remark 2.40. Clearly every front arises as an F^h for some locally constant multi-sequence h. Indeed for any front F and any injective super-sequence f from F, we have $F = F^{f^\uparrow}$. Therefore we can think of the definition of a front as a characterisation of those families of finite subsets of ω arising as an F^h for some locally constant multi-sequence h.

The basic properties of the correspondence $h \mapsto h^\downarrow$ and $f \mapsto f^\uparrow$ are easily stated with the help of the following partial order among super-sequences into a given set.

Definition 2.41. Let both F and G be fronts on the same set $X \in [\omega]^\infty$ and $f : F \to E$ and $g : G \to E$ be any maps. We write $f \sqsubseteq g$ if $F \subseteq \overline{G}$ and for every $s \in F$ and every $t \in G$, $s \sqsubseteq t$ implies $f(s) = g(t)$.

To simplify notation we write $\check{f} : \check{F} \to E$ instead of $(f^\uparrow)^\downarrow : F^{f^\uparrow} \to E$.

Fact 2.42. *Let $X \in [\omega]^\infty$ and E be a set.*

(i) *for every front F on X and every map $f : F \to E$, the map $\check{f} : \check{F} \to E$ is such that $\check{f} \sqsubseteq f$.*

(ii) *for every fronts F and G on X and maps $f : F \to E$ and $g : G \to E$, $f \sqsubseteq g$ implies $f^{\uparrow} = g^{\uparrow}$.*

(iii) *for every locally constant map $h : [X]^{\infty} \to E$, we have $(h^{\downarrow})^{\uparrow} = h$.*

It follows that for every locally constant multi-sequence $h : [X]^{\infty} \to E$ the super-sequence $h^{\downarrow} : F^{h} \to E$ is the minimal element for $\sqsubseteq$ among the set of super-sequences $g : G \to E$ with $g^{\uparrow} = h$. Moreover for every super-sequence $f : F \to E$ the super-sequence $\check{f} : \check{F} \to E$ is the $\sqsubseteq$-minimal among the super-sequences g with $g \sqsubseteq f$. In particular $\check{\check{f}} = \check{f}$ for every super-sequence f.

The super-sequences which are $\sqsubseteq$-minimal will play a rôle later and deserve a name.

Definition 2.43. Let E be a set and F a front on X. A super-sequence $f : F \to E$ is said to be *spare* if f is minimal for $\sqsubseteq$, or equivalently $\check{f} = f$, i.e. if $\check{F} = F$.

Example 2.44. If $c : F \to E$ is constant equal to $e \in E$ then c is not spare and of course $\check{c} : \{\emptyset\} \to E$, $\emptyset \mapsto e$.

The following is a simple characterisation of spare super-sequences.

Lemma 2.45. *Let $f : F \to E$ be a map from a front to some set E. Then the following are equivalent*

(i) *f is spare,*

(ii) *for every $s \in \overline{F}$, if for every $t, t' \in F$ $s \sqsubseteq t$ and $s \sqsubseteq t'$ imply $f(t) = f(t')$, then $s \in F$.*

Proof. Suppose that $s \in \overline{F} \setminus F$ is such that for every $t, t' \in F$, $s \sqsubseteq t$ and $s \sqsubseteq t'$ imply $f(t) = f(t')$. Then $\hat{f}$ is constant on M_s but $s \sqsubset t \in F$ so $t \notin F^{\hat{f}}$. Therefore f is not spare.

Conversely if f is not spare, then there exists $t \in F \setminus \check{F}$. This means that there is $s \in \check{F}$ with $s \sqsubset t$ and $f^{\uparrow}$ is constant on M_s, so for every $t' \in F$ with $s \sqsubseteq t'$ we have $f(t) = f(t')$. □

2.2.4 Iterated powerset, determinacy of finite games

> *It also transpires that if, by a certain fairly natural extension of our definition of $[\mathcal{P}^n(Q)]$, we define $[\mathcal{P}^{\alpha}(Q)]$ for every ordinal α, then Q is bqo iff $[\mathcal{P}^{\alpha}(Q)]$ is wqo for every ordinal α. To justify these statements would not be relevant here, but it was from this point of view that the author was first led to study bqo sets.*

Crispin St. John Alvah Nash-Williams [Nas65, p. 700]

Following in Nash-Williams' steps, we introduce the notion of better-quasi-orders as the quasi-orders whose iterated powersets are WQO. We do this in the light of further developments of the theory, taking advantage of Simpson's point of view on super-sequences, using the determinacy of finite games and a powerful game-theoretic technique invented by Tony Martin.

First let us define precisely the iterated powerset of a qo together with its lifted quasi-order. To facilitate the following discussion we focus on the non-empty sets over some quasi-order Q. Let $\mathcal{P}^*(A)$ denote the set of non-empty subsets of a set A, i.e. $\mathcal{P}^*(A) = \mathcal{P}(A) \setminus \{\emptyset\}$. We define by transfinite recursion

$$\begin{aligned} V_0^*(Q) &= Q \\ V_{\alpha+1}^*(Q) &= \mathcal{P}^*(V_\alpha^*(Q)) \\ V_\lambda^*(Q) &= \bigcup_{\alpha<\lambda} V_\alpha^*(Q), \quad \text{for } \lambda \text{ limit.} \end{aligned}$$

We treat the element of Q as *urelements* or *atoms*, namely they have no elements but they are different from the empty set. Let

$$V^*(Q) = \bigcup_\alpha V_\alpha^*(Q).$$

Let us say that a set A is Q-transitive if for all $x \in A$ we have $x \subseteq A$ whenever $x \notin Q$. We define the *Q-transitive closure* of $X \in V^*(Q)$, denoted by $\mathrm{tc_Q}(X)$, as the smallest Q-transitive set containing X. Notice that in particular $\mathrm{tc_Q}(q) = \{q\}$ for every $q \in Q$ and that $\mathrm{tc_Q}(X) = X$ for every non-empty subset X of Q. Finally we define for every $X \in V^*(Q)$ the *support* of X, denoted by $\mathrm{supp_Q}(X)$, as the set of elements of Q which belong to $\mathrm{tc_Q}(X)$:

$$\mathrm{supp_Q}(X) = Q \cap \mathrm{tc_Q}(X).$$

Notice that $\mathrm{supp_Q}(X)$ is never empty and that for every $q \in Q$ we have $\mathrm{supp_Q}(q) = \{q\}$.

Following an idea of Forster [For03] we define the quasi-order on $V^*(Q)$ via the existence of a winning strategy in a natural game. We refer the reader to Kechris [Kec95, (20.)] for the basic definitions pertaining to two-player games with perfect information.

Definition 2.46. For every $X, Y \in V^*(Q)$ we define a two-player game with perfect information $G_{V^*}(X, Y)$ by induction on the membership relation. The game $G_{V^*}(X, Y)$ goes as follows. Player I starts by choosing some X' such that:

- if $X \notin Q$, then $X' \in X$,
- otherwise, $X' = X$.

Then Player II replies by choosing some Y' such that:

- if $Y \notin Q$, then $Y' \in Y$,
- otherwise $Y' = Y$.

If both X' and Y' belong to Q, then Player II wins if $X' \leqslant Y'$ in Q and Player I wins if $X' \nleqslant Y'$. Otherwise the game continues as in $G_{V^*}(X', Y')$.

We then define the lifted quasi-order on $V^*(Q)$ by letting for $X, Y \in V^*(Q)$

$$X \leqslant Y \quad \longleftrightarrow \quad \text{Player II has a winning strategy in } G_{V^*}(X, Y).$$

Remark 2.47. The above definition can be rephrased by induction on the membership relation as follows:

(1) if $X, Y \in Q$, then $X \leqslant Y$ if and only if $X \leqslant Y$ in Q,

(2) if $X \in Q$ and $Y \notin Q$, then

$$X \leqslant Y \quad \longleftrightarrow \quad \text{there exists } Y' \in Y \text{ with } X \leqslant Y',$$

(3) if $X \notin Q$ and $Y \in Q$, then

$$X \leqslant Y \quad \longleftrightarrow \quad \text{for every } X' \in X \text{ we have } X' \leqslant Y,$$

(4) if $X \notin Q$ and $Y \notin Q$, then

$$X \leqslant Y \quad \longleftrightarrow \quad \text{for every } X' \in X \text{ there exists } Y' \in Y \text{ with } X' \leqslant Y'.$$

Our definition coincides with the one given by Shelah [She82, Claim 1.7, p.188]. But Milner [Mil85] and Laver [Lav71] both omit condition (3).

The axiom of foundation ensures that in any play of a game $G_{V^*}(X, Y)$ a round where both players have chosen elements of Q is eventually reached, resulting in the victory of one of the two players. In particular, each game $G_{V^*}(X, Y)$ is determined as already proved by Von Neumann and Morgenstern [VM44] (see [Kec95, (20.1)]). The crucial advantage of the game-theoretic formulation of the quasi-order on $V^*(Q)$ resides in the fact that the negative

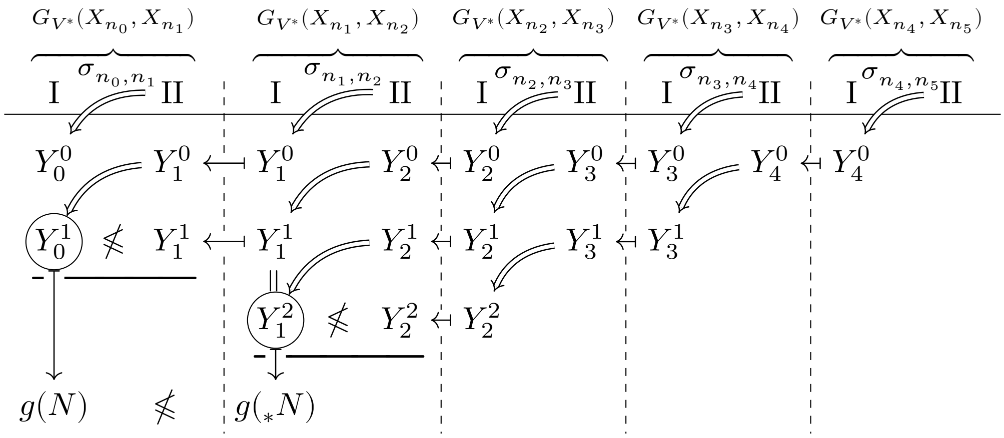


Figure 2.3: Constructing a multi-sequence by stringing strategies together.

condition $X \not\leq Y$ is equivalent to the existential statement 'Player I has a winning strategy'.

Now suppose Q is a quasi-order such that $V^*(Q)$ is not WQO and let $(X_n)_{n\in\omega}$ be a bad sequence in $V^*(Q)$. Whenever $m < n$ we have $X_m \not\leq X_n$ and we can choose a winning strategy $\sigma_{m,n}$ for Player I in $G_{V^*}(X_m, X_n)$. We define a locally constant multi-sequence $g : [\omega]^\infty \to Q$ as follows. Let $N = \{n_0, n_1, n_2, ...\}$ be an infinite subset of ω enumerated in increasing order. We define $g(N)$ as the last move of Player I in a particular play of $G_{V^*}(X_{n_0}, X_{n_1})$ in a way best understood by contemplating Figure 2.3.

Let Y_0^0 be the the first move of Player I in $G_{V^*}(X_{n_0}, X_{n_1})$ as prescribed by its winning strategy σ_{n_0,n_1}. Then let Player II copy the first move Y_1^0 of Player I given by the strategy σ_{n_1,n_2} in $G_{V^*}(X_{n_1}, X_{n_2})$. Then Player I answers Y_0^1 according to the strategy σ_{n_0,n_1}. Now if Y_1^0 is not in Q, then we need to continue our play of $G_{V^*}(X_{n_1}, X_{n_2})$ a little further to determine the second move of Player II in $G_{V^*}(X_{n_0}, X_{n_1})$. Let the first move of Player II in $G_{V^*}(X_{n_1}, X_{n_2})$ be the first move of Player I in $G_{V^*}(X_{n_2}, X_{n_3})$ as prescribed by his winning strategy σ_{n_2,n_3}. Then this determines the second move Y_1^1 of Player I in $G_{V^*}(X_{n_1}, X_{n_2})$ according to σ_{n_1,n_2}. We then let the second move of Player II in $G_{V^*}(X_{n_0}, X_{n_1})$ to be this Y_1^1. This yields some answer Y_0^1 of Player I according to σ_{n_0,n_1}. We continue so on and so forth until the play of $G_{V^*}(X_{n_0}, X_{n_1})$ reaches an end with some $(Y_0^{k_N}, Y_1^{k_N}) \in Q \times Q$ and we let $g(N) = Y_0^{k_N}$. Since the play of $G_{V^*}(X_{n_0}, X_{n_1})$ is finite, $g(N)$ depends only on a finite initial segment of N and we have therefore defined a locally constant multi-sequence $g : [\omega]^\infty \to Q$.

Now since Player I has followed the winning strategy σ_{n_0,n_1} we have $Y_0^{k_N} \nleqslant Y_1^{k_N}$. Now if the play of the game $G_{V^*}(X_{n_1}, X_{n_2})$ has not yet reached an end at step k_N we go on in the same fashion. Assume it ends with some pair (Y_1^l, Y_2^l) in Q. By the rules of the game G_{V^*}, since $Y_1^{k_N} \in Q$ we necessarily have $Y_1^l = Y_1^{k_N}$. But Y_1^l is just $g(\{n_1, n_2, n_3, ...\})$, hence for every $N \in [\omega]^\infty$ we have

$$g(N) \nleqslant g(N \setminus \{\min N\}).$$

For every $N \in [\omega]^\infty$ we call the *shift of* N, denoted by ${}_*N$, the set $N \setminus \{\min N\}$. We are led to the following:

Definition 2.48. Let Q be a qo and $h : [X]^\infty \to Q$ a multi-sequence.

(1) We say that h is *bad* if $h(N) \nleqslant h({}_*N)$ for every $N \in [X]^\infty$,

(2) We say that h is *good* if there exists $N \in [X]^\infty$ with $h(N) \leqslant h({}_*N)$,

At last, we present the deep definition due to Nash-Williams here in a modern reformulation.

Definition 2.49. A quasi-order Q is a *better-quasi-order* (BQO) if there is no bad locally constant multi-sequence into Q.

Of course the definition of better-quasi-order can be formulated in terms of super-sequences as Nash-Williams originally did. The only missing ingredient is a counterpart of the shift map $N \mapsto {}_*N$ on finite subsets of natural numbers.

Definition 2.50. For $s, t \in [\omega]^{<\infty}$ we say that t is a shift of s and write $s \lhd t$ if there exists $X \in [\omega]^\infty$ such that

$$s \sqsubset X \text{ and } t \sqsubset {}_*X.$$

Definitions 2.51. Let Q be a qo and $f : F \to Q$ be a super-sequence.

(1) We say that f is *bad* if whenever $s \lhd t$ in F, we have $f(s) \nleqslant f(t)$.

(2) We say that f is *good* if there exists $s, t \in F$ with $s \lhd t$ and $f(s) \leqslant f(t)$.

Lemma 2.52. *Let Q be a quasi-order.*

(i) *If $h : [\omega]^\infty \to Q$ is locally constant and bad, then $h^\downarrow : F^h \to Q$ is a bad super-sequence.*

(ii) *If $f : F \to Q$ is a bad super-sequence from a front on X, then $f^{\uparrow} : [X]^{\infty} \to Q$ is a bad locally constant multi-sequence.*

Proof. (i) Suppose $h : [X]^{\infty} \to Q$ is locally constant and bad. Let us show that $h^{\downarrow} : F^{h} \to Q$ is bad. If $s, t \in F^{h}$ with $s \lhd t$, i.e. there exists $Y \in [X]^{\infty}$ such that $s \sqsubset Y$ and $t \sqsubset {}_{*}Y$. Then $h^{\downarrow}(s) = h(Y)$ and $h^{\downarrow}(t) = h({}_{*}Y)$ and since h is assumed to be bad, we have $h^{\downarrow}(s) \nleq h^{\downarrow}(t)$.

(ii) Suppose $f : F \to Q$ is bad from a front on X and let $Y \in [X]^{\infty}$. There are unique $s, t \in F$ such that $s \sqsubset Y$ and $t \sqsubset {}_{*}Y$, and clearly $f^{\uparrow}(Y) = f(s)$, $f^{\uparrow}({}_{*}Y) = f(t)$, and $s \lhd t$. Therefore $f^{\uparrow}(X) \nleq f^{\uparrow}({}_{*}X)$ holds. □

Proposition 2.53. *For a quasi-order Q the following are equivalent.*

(i) *Q is a better-quasi-order,*

(ii) *there is no bad super-sequence into Q,*

(iii) *there is no bad spare super-sequence into Q.*

The idea of stringing strategies together that we used to arrive at the definition of BQO is directly inspired from a famous technique used by van Engelen, Miller, and Steel [vEMS87, Theorem 3.2] together with Louveau and Saint Raymond [LS90, Theorem 3]. This method was first applied by Martin in the proof of the well-foundedness of the Wadge hierarchy (see [Kec95, (21.15), p. 158]). Forster [For03] introduces better-quasi-orders in a very similar way, but a super-sequence instead of a multi-sequence is constructed, making the similarity with the method used by van Engelen, Miller, and Steel [vEMS87] and Louveau and Saint Raymond [LS90] less obvious. One of the advantages of multi-sequences resides in the fact that they enable us to work with super-sequences without explicitly referring to their domains. This is particularly useful in the above construction, since a bad sequence in $V^{*}(Q)$ can yield a multi-sequence whose underlying front is of arbitrarily large rank. Indeed Marcone [Mar94] showed that super-sequences from fronts of arbitrarily large rank are required in the definition of BQO.

Notice that the notion of BQO naturally lies between those of well-orders and WQO.

Proposition 2.54. *Let Q be a qo. Then*

$$Q \text{ is a well-order} \quad \to \quad Q \text{ is BQO} \quad \to \quad Q \text{ is WQO}.$$

Proof. Suppose Q is a well order and let $h : [\omega]^\infty \to Q$ is any multi-sequence into Q. Fix $X \in [\omega]^\infty$ and let $X_0 = X$ and $X_{n+1} = {}_*X_n$. Since Q is a well-order, there exists n such that $h(X_n) \leqslant h(x_{n+1})$, otherwise $h(X_n)$ would be a descending chain in Q. So h is good and therefore Q is BQO.

Now observe that for $m, n \in \{\omega\}$ we have $\{m\} \lhd \{n\}$ if and only if $m < n$. So if Q is BQO, then in particular every sequence $f : [\omega]^1 \to Q$ is good, and so Q is WQO. □

2.2.5 Equivalence

Pushing further the idea that led us to the definition of BQO, we can build from any bad multi-sequence in $V^*(Q)$ a bad multi-sequence in Q. Therefore proving that if Q is BQO, then $V^*(Q)$ is actually BQO.

Proposition 2.55. *Let Q be a qo. For every bad locally constant $h : [\omega]^\infty \to V^*(Q)$ there exists a bad locally constant $g : [\omega]^\infty \to Q$ such that $g(X) \in \mathrm{supp}_Q(h(X))$ for every $X \in [\omega]^\infty$.*

Proof. Let $h : [\omega]^\infty \to V^*(Q)$ be locally constant and bad, and let us write $h(X) = h_X$ for $X \in [\omega]^\infty$. Notice that the image of h is countable and choose for every $X \in [\omega]^\infty$ a winning strategy σ_X for Player I in $G_{V^*}(h_X, h_{{}_*X})$. We let $X_0 = X$ and $X_{n+1} = {}_*X_n$.

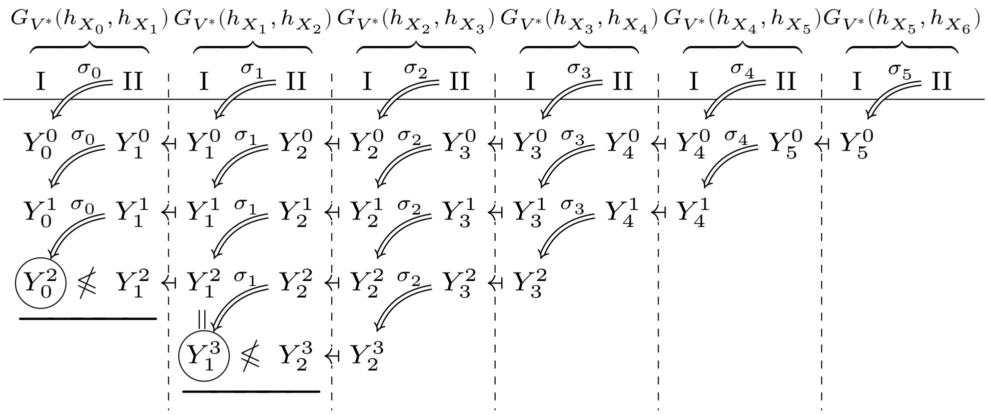


Figure 2.4: Stringing strategies together.

Consider the diagram in Figure 2.4 obtained by letting Player I follow the winning strategy $\sigma_n = \sigma_{X_n}$ in $G_{V^*}(h_{X_n}, h_{X_{n+1}})$ and II responding in $G_{V^*}(h_{X_n}, h_{X_{n+1}})$ by copying I's moves in $G_{V^*}(h_{X_{n+1}}, h_{X_{n+2}})$. This uniquely determines for each

n a finite play $(Y^i_n, Y^i_{n+1})_{i\leqslant l_n}$ of the game $G_{V^*}(h_{X_n}, h_{X_{n+1}})$ ending with some $Y^{l_n}_n \nleqslant Y^{l_n}_{n+1}$ in Q. Clearly the play $(Y^i_n, Y^i_{n+1})_{i\leqslant l_n}$ depends only on the value taken by h on the X_j with $j \in \{n, \dots, n+l_n+2\}$. By the rules of the game G_{V^*} for every n we have $Y^{l_n}_{n+1} = Y^{l_{n+1}}_{n+1}$. We let $Y^X_0 = Y^{l_0}_0$ and $Y^X_{n+1} = Y^{l_n}_{n+1} = Y^{l_{n+1}}_{n+1}$. We define $g : [\omega]^\infty \to Q$ by letting $g(X) = Y^X_0$. Since Y^X_0 depends only on $h_{X_0}, \dots h_{X_{l_0+2}}$ and h is locally constant, it follows that g is locally constant. Moreover, by construction $g({}_*X) = Y^{{}_*X}_0 = Y^X_1$ and so $g(X) \nleqslant g({}_*X)$. □

Corollary 2.56. *If Q is* BQO*, then $V^*(Q)$ is* BQO*.*

We now briefly show that there is a strong converse to Corollary 2.56.

Let $f : F \to Q$ be a super-sequence from a front on ω into a qo Q. We define by recursion on the well-founded relation $\sqsupset$ on $\overline{F}$ a map $\tilde{f} : \overline{F} \to V^*(Q)$ by

$$\begin{aligned} \tilde{f}(s) &= f(s) && \text{if } s \in F, \\ \tilde{f}(s) &= \{\tilde{f}(s \cup \{n\}) \mid n \in \omega/s \text{ and } s \cup \{n\} \in \overline{F}\} && \text{otherwise.} \end{aligned}$$

As long as F is not trivial we have $[\omega]^1 \subseteq \overline{F}$ and restricting $\tilde{f}$ to $[\omega]^1$ we obtain the sequence $\tilde{f}\restriction_{[\omega]^1} : [\omega]^1 \to V^*(Q)$. Notice also that $\tilde{f}(s) \in Q$ if and only if $s \in F$.

Lemma 2.57. *If $f : F \to Q$ is bad, then $\tilde{f}\restriction_{[\omega]^1}$ is a bad sequence in $V^*(Q)$.*

Proof. By way of contradiction suppose that for some $m_0, n_0 \in \omega$ with $m_0 < n_0$ we have $\tilde{f}(m_0) \leqslant \tilde{f}(n_0)$ in $V^*(Q)$ and let σ be a winning strategy for Player II in $G_{V^*}(\tilde{f}(m_0), \tilde{f}(n_0))$. Let $s_0 = (m_0)$, $t_0 = (n_0)$ and $u_0 = (m_0, n_0)$. We consider the following play of $G_{V^*}(\tilde{f}(m_0), \tilde{f}(n_0))$. Observe that if $s_0 = (m_0) \notin F$, then $u_0 = (m_0, n_0) \in \overline{F}$. We make Player I start with $\tilde{f}(s_1)$ where $s_1 = s_0$ if $s_0 \in F$ and $s_1 = u_0$ otherwise. Then II answers according to σ by $\tilde{f}(t_1)$ for some $t_1 \in \overline{F}$. If $t_0 = (n_0) \in F$, then necessarily $t_1 = t_0$ and we let $u_1 = u_0 \frown (k)$ with $k = 1 + \max u_0$. Otherwise $t_0 \sqsubset t_1$ and $t_1 = (n_0, n_1)$ for some $n_1 > n_0$, we then let $u_1 = u_0 \cup t_1 = u_0 \frown (n_1)$. Notice that in any case $s_1 \lhd t_1$ since for $X = u_1 \cup \omega/u_1$ we have $s_1 \sqsubset X$ and $t_1 \sqsubset {}_*X$. Then we make I respond with $\tilde{f}(s_2)$ where $s_2 = s_1$ if $s_1 \in F$, $s_2 = u_2$ if $s_1 \notin F$. We continue in this fashion, an example of which is depicted in Figure 2.5. After finitely many rounds I has reached some $f(s)$ for $s \in F$, and II has reached some $f(t)$ with $t \in F$. By construction $s \lhd t$, but since σ is winning for II, we have $f(s) \leqslant f(t)$, a contradiction. □

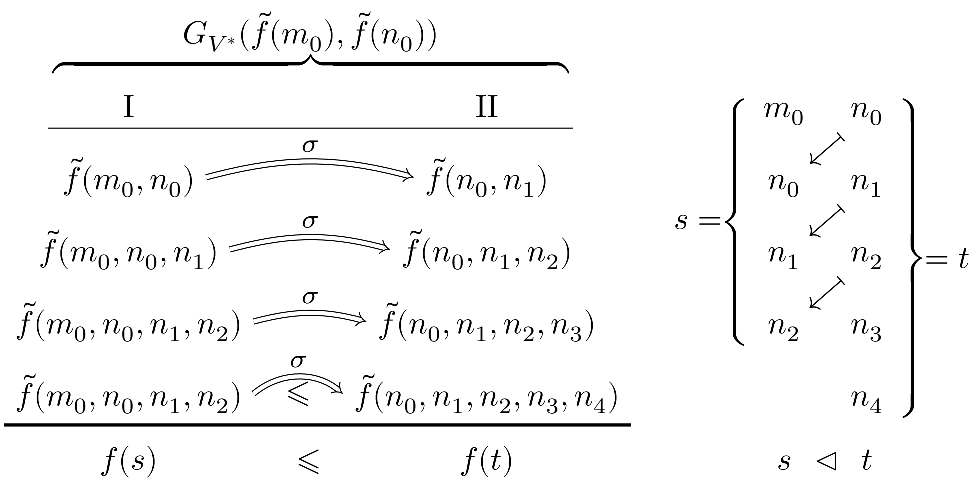


Figure 2.5: Copying and shift.

Notice that by definition $\tilde f : \overline{F} \to V^*(Q)$ only reaches *hereditarily countable non-empty sets over* Q, namely the elements of Q and the countable non-empty sets of hereditarily countable non-empty sets over Q. Let $H^*_{\omega_1}(Q)$ denote the set of hereditarily countable non-empty sets over Q equipped with the qo induced from $V^*(Q)$. By the axiom of countable choice we have $X \in H^*_{\omega_1}(Q)$ if and only if $X \in V^*(Q)$ and the set $\mathrm{tc}_{\mathrm{Q}}(X)$ is countable. We have obtained the well known equivalence.

Theorem 2.58. *A quasi-order Q is* BQO *if and only if $H^*_{\omega_1}(Q)$ is* WQO.

Notice that by definition any countable non-empty subset of $H^*_{\omega_1}(Q)$ belongs to $H^*_{\omega_1}(Q)$. Moreover, by Proposition 2.6 (W6) a quasi-order is WQO if and only if the qo $\mathcal{P}_{<\aleph_1}(Q)$ of its countable subsets is well-founded, so $H^*_{\omega_1}(Q)$ is WQO if and only if it is well-founded.

Theorem 2.59. *A quasi-order Q is* BQO *if and only if $H^*_{\omega_1}(Q)$ is well-founded.*

2.3 Around the definition of better-quasi-order

In the previous section, we were led to the definition of BQOs by reflecting a bad sequence in $V^*(Q)$ into some bad multi-sequence in Q. In this section, we discuss the definition we obtained and try to understand what its essential features are. Along this line we show that the presence of the shift is somewhat accidental. The material of this section has not yet been published by the author.

2.3.1 The perfect versus bad dichotomy

For every $X \in [\omega]^\infty$, let us denote invariably by $\mathsf{S} : [X]^\infty \to [X]^\infty$ the *shift map* defined by $\mathsf{S}(N) = {}_*N$ for every $N \in [X]^\infty$.

For our discussion, we wish to treat both the pairs $([X]^\infty, \mathsf{S})$, for $X \in [\omega]^\infty$, and the quasi-orders $(Q, \leqslant)$ as objects in the same category.

With this in mind, let us call a *topological digraph* a pair (A, R) consisting of a topological space A together with a binary relation R on A. If (A, R) and (B, S) are topological digraphs, a *continuous homomorphism* from (A, R) to (B, S) is a continuous map $\varphi : A \to B$ such that for every $a, a' \in A$, $a \; R \; a'$ implies $\varphi(a) \; S \; \varphi(a')$. As an important particular case, if $f : A \to A$ is any function we write (A, f) for the topological digraph whose binary relation is the graph of the function f. If $f : A \to A$ and $g : B \to B$ are functions, a map $\varphi : A \to B$ is a continuous homomorphism from (A, f) to (B, g) exactly in case φ is continuous and $\varphi \circ f = g \circ \varphi$. For a binary relation R on A let us denote by $R^\complement$ the binary relation $(A \times A) \setminus R$.

Observe that for a discrete space A, a multi-sequence $h : [X]^\infty \to A$ is continuous exactly when it is locally constant.

Proposition 2.60. *Let $f : [\omega]^\infty \to [\omega]^\infty$ be a continuous map such that $f(X) \subseteq X$ for every $X \in [\omega]^\infty$ and R be a binary relation on a discrete space A. For every continuous $\varphi : [\omega]^\infty \to A$ there exists $Z \in [\omega]^\infty$ such that*

either $\varphi : ([Z]^\infty, f) \to (A, R)$ *is a continuous homomorphism,*

or $\varphi : ([Z]^\infty, f) \to (A, R^\complement)$ *is a continuous homomorphism.*

Proof. Let $\varphi : [\omega]^\infty \to (A, R)$ be locally constant and define $c : [\omega]^\infty \to 2$ by $c(X) = 1$ if and only if $\varphi(X) \; R \; \varphi(f(X))$. Clearly c is locally constant so let $c^\downarrow : F^c \to 2$ be the associated super-sequence. By Nash-Williams' Theorem 2.36 there exists an infinite subset Z of ω such that $c^\downarrow\restriction_{F^c|Z} : F^c|Z \to 2$ is constant. Therefore for the restriction $\psi = \varphi\restriction_{[Z]^\infty} : [Z]^\infty \to A$ it follows that either $\psi : ([Z]^\infty, f) \to (A, R^\complement)$ is a continuous homomorphism, or $\psi : ([Z]^\infty, f) \to (A, R)$ is a continuous homomorphism. □

Remark 2.61. The previous proposition generalises as follows. Let A be any topological space, $R \subseteq A \times A$ be a Borel binary relation and $f : [\omega]^\infty \to [\omega]^\infty$ a Borel map such that $f(X) \subseteq X$ for every $X \in [\omega]^\infty$. For every Borel map $\varphi : [\omega]^\infty \to A$ there exists $Z \in [\omega]^\infty$ such that either $\varphi : ([Z]^\infty, f) \to (A, R)$ is a Borel homomorphism,

or $\varphi : ([Z]^\infty, f) \to (A, R^\complement)$ is a Borel homomorphism.

Indeed, the set

$$\{X \in [\omega]^\infty \mid \varphi(X) \; R \; \varphi(f(X))\} = (\varphi \times (\varphi \circ f))^{-1}(R)$$

is Borel in $[\omega]^\infty$ and thus, by the Galvin-Prikry theorem [GP73], there exists a $Z \in [\omega]^\infty$ as required.

Definition 2.62. Let R be a binary relation on a discrete space A.

(1) A multi-sequence $h : [X]^\infty \to A$ is *perfect* if $h : ([X]^\infty, \mathsf{S}) \to (A, R)$ is a homomorphism, i.e. if $h(N) \; R \; h({}_*N)$ for every $N \in [X]^\infty$,

(2) A super-sequence $f : F \to A$ is *perfect* if for every $s, t \in F$, $s \lhd t$ implies $f(s) \; R \; f(t)$.

In particular letting $f = \mathsf{S}$ in Proposition 2.60, we obtain the following well-known equivalence.

Corollary 2.63. *For a quasi-order Q the following are equivalent.*

(i) *Q is* BQO,

(ii) *every locally constant multi-sequence in Q admits a perfect sub-multi-sequence.*

(iii) *every super-sequence in Q admits a perfect sub-super-sequence.*

Proposition 2.60 also suggests the following generalisation of the notion of BQO to arbitrary relations:

Definition 2.64. A binary relation R on a discrete space A is a *better-relation* on A if there is no continuous homomorphism $\varphi : ([\omega]^\infty, \mathsf{S}) \to (A, R^\complement)$.

This definition first appeared in a paper by Shelah [She82] and plays an important rôle in a work by Marcone [Mar94]. Of course a better-quasi-order is simply a better-relation which is reflexive and transitive.

Remark 2.65. One could also consider non discrete analogues of the notion of better-quasi-orders and better-relations. Louveau and Saint Raymond [LS90] define a *topological better-quasi-order* as a pair $(A, \leqslant)$, where A is a topological space and $\leqslant$ is a quasi-order on A, such that there is no Borel homomorphism $\varphi : ([\omega]^\infty, \mathsf{S}) \to (A, \leqslant^\complement)$. We believe that topological analogs of BQO and better-relations deserve further investigations.

2.3.2 Generalised shifts

The topological digraph $([\omega]^\infty, \mathsf{S})$ is central to the definition of BQO. Indeed a qo Q is BQO if and only if there is no continuous morphism $h : ([\omega]^\infty, \mathsf{S}) \to (Q, \leqslant^{\complement})$. In general, one can ask for the following:

Problem 1. *Characterise the topological digraphs which can be substituted for* $([\omega]^\infty, \mathsf{S})$ *in the definition of* BQO*.*

Let us write $(A, R) \leqslant_{\mathrm{ch}} (B, S)$ if there exists a continuous homomorphism from (A, R) to (B, S) and $(A, R) \equiv_{\mathrm{ch}} (B, S)$ if both $(A, R) \leqslant_{\mathrm{ch}} (B, S)$ and $(B, S) \leqslant_{\mathrm{ch}} (A, R)$ hold.

Notice that a binary relation S on a discrete space B is a better-relation if and only if $([\omega]^\infty, \mathsf{S}) \not\leqslant_{\mathrm{ch}} (B, S^{\complement})$. Therefore any topological digraph (A, R) with $(A, R) \equiv_{\mathrm{ch}} ([\omega]^\infty, \mathsf{S})$ can be used in the definition of better-relation in place of $([\omega]^\infty, \mathsf{S})$. We do not know whether the converse holds, namely if (A, R) is a topological digraph which can be substituted to $([\omega]^\infty, \mathsf{S})$ in the definition of BQO, does it follow that $(A, R) \equiv_{\mathrm{ch}} ([\omega]^\infty, \mathsf{S})$?

We now show that at least the shift map S can be replaced by certain 'generalised shifts'. To this end, we first observe that the topological space $[\omega]^\infty$ admits a natural structure of monoid. Following Solecki [Sol13] and Prömel and Voigt [PV86], we use the language of increasing injections rather than that of sets. We denote by $\mathcal{E}$ the monoid of embeddings of $(\omega, <)$ into itself under composition,

$$\mathcal{E} = \{f : \omega \to \omega \mid f \text{ is injective and increasing}\}.$$

For every $X \in [\omega]^\infty$, we let $f_X \in \mathcal{E}$ denote the unique increasing and injective enumeration of X. Conversely we associate to each $f \in \mathcal{E}$ the infinite subset $\operatorname{Im} f$ of ω consisting in the range of f. Therefore the set of substructures of $(\omega, <)$ which are isomorphic to the whole structure $(\omega, <)$, namely $[\omega]^\infty$, is in one-to-one correspondence with the monoid of embeddings of $(\omega, <)$ into itself. Moreover observe that for all $X, Y \in [\omega]^\infty$ we have

$$X \subseteq Y \quad \longleftrightarrow \quad \exists g \in \mathcal{E} \quad f_X = f_Y \circ g,$$

so the inclusion relation on $[\omega]^\infty$ is naturally expressed in terms of the monoid operation. Also, the set $[X]^\infty$ corresponds naturally to the following right ideal:

$$f_X \circ \mathcal{E} = \{f_X \circ g \mid g \in \mathcal{E}\}.$$

As for $[\omega]^\infty$, $\mathcal{E}$ is equipped with the topology induced by the Baire space ω^ω of all functions from ω to ω. In particular, the composition $\circ : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$, $(f, g) \mapsto f \circ g$ is continuous for this topology.

Observe now that, in the terminology of increasing injections, the shift map $\mathsf{S} : \mathcal{E} \to \mathcal{E}$ is simply the composition on the right with the *successor function* $\mathsf{s} \in \mathcal{E}$, $\mathsf{s}(n) = n + 1$. Indeed for every X

$$f_{{}_*X} = f_X \circ \mathsf{s}.$$

This suggests to consider arbitrary injective increasing function g, $g \neq \mathrm{id}_\omega$, in place of the successor function. For any $g \in \mathcal{E}$, we write $\vec{g} : \mathcal{E} \to \mathcal{E}$, $f \mapsto f \circ g$ for the composition on the right by g. In particular, $\vec{\mathsf{s}} = \mathsf{S}$ is the usual shift and in our new terminology we have $([\omega]^\infty, \mathsf{S}) = (\mathcal{E}, \vec{\mathsf{s}})$.

The main result of this section is that these generalised shifts $\vec{g}$ are all equivalent as far as the theory of better-relations is concerned.

Theorem 2.66. *For every increasing injective function $g \in \mathcal{E}$, with $g \neq \mathrm{id}_\omega$, we have $(\mathcal{E}, \vec{g}) \equiv_{ch} ([\omega]^\infty, \mathsf{S})$.*

Theorem 2.66 follows from Lemmas 2.72 and 2.73 below, but let us first state explicitly some of the direct consequences.

Remark 2.67. Every topological digraph (A, R) has an associated topological graph (A, R^{s}) whose symmetric and irreflexive relation R^{s} is given by

$$a\ R^{\mathrm{s}}\ b \quad \longleftrightarrow \quad a \neq b \text{ and } (a\ R\ b \text{ or } b\ R\ a).$$

The chromatic number and Borel chromatic number of topological graphs are studied by Kechris, Solecki, and Todorčević [KST99]. Notably the associated graph of $([\omega]^\infty, \mathsf{S})$ has chromatic number 2 and Borel chromatic number $\aleph_0$ (see also the paper by Di Prisco and Todorčević [DT06]). It directly follows from Theorem 2.66 that for every $g \in \mathcal{E}$, with $g \neq \mathrm{id}_\omega$, the associated graph of $(\mathcal{E}, \vec{g})$ also has chromatic number 2 and Borel chromatic number $\aleph_0$.

Definition 2.68. Let $g \in \mathcal{E}$, R a binary relation on a discrete space A. We say (A, R) is a *g-better-relation* if one of the following equivalent conditions hold:

(1) for every continuous $\varphi : \mathcal{E} \to A$ there exists $f \in \mathcal{E}$ such that the restriction $\varphi_f : (f \circ \mathcal{E}, \vec{g}) \to (A, R)$ is a continuous morphism,

(2) there is no continuous morphism $\varphi : (\mathcal{E}, \vec{g}) \to (A, R^{\complement})$.

In case $\leqslant$ is a quasi-order on a discrete space Q, we say that Q is g-BQO instead of $(Q, \leqslant)$ is a g-better-relation.

Of course this notion trivialises for $g = \mathrm{id}_\omega$, since an id_ω-better-relation is simply a reflexive relation. Moreover better relation corresponds to s-better-relation.

Theorem 2.69. *Let $g \in \mathcal{E} \setminus \{\mathrm{id}_\omega\}$, R a binary relation on a discrete space A. Then R is a g-better-relation if and only if R is a better-relation. In particular, a quasi-order $(Q, \leqslant)$ is g-BQO if and only if $(Q, \leqslant)$ is BQO.*

Corollary 2.70. *A qo Q is BQO if and only if for every locally constant $\varphi : \mathcal{E} \to Q$ and every $g \in \mathcal{E}$ there exists $f \in \mathcal{E}$ such that*

$$\varphi(f) \leqslant \varphi(f \circ g).$$

As a corollary we have the following strengthening of Corollary 2.63 which is obtained by repeated applications of Proposition 2.60.

Proposition 2.71. *Let Q be BQO and $\varphi : \mathcal{E} \to Q$ be locally constant. For every finite subset $\mathcal{G}$ of $\mathcal{E}$ there exists $h \in \mathcal{E}$ such that the restriction $\varphi : h \circ \mathcal{E} \to Q$ is perfect with respect to every member of $\mathcal{G}$, i.e. for every $f \in \mathcal{E}$ and every $g \in \mathcal{G}$*

$$\varphi(h \circ f) \leqslant \varphi(h \circ f \circ g).$$

Getting a result of this kind was one of our motivations for proving Theorem 2.66. Indeed they are situations where we need to deal with an arbitrary super-sequence into some BQO and where going to a perfect sub-super-sequence is not enough (see Remark 4.44). We believe that Proposition 2.71 can still be improved.

Finally here are the two lemmas which yield the proof of Theorem 2.66.

Lemma 2.72. *Let $g \in \mathcal{E} \setminus \{\mathrm{id}_\omega\}$. Then $(\mathcal{E}, \vec{g}) \leqslant_{ch} (\mathcal{E}, \vec{\mathsf{s}})$, i.e. there exists a continuous map $\rho : \mathcal{E} \to \mathcal{E}$ such that for every $f \in \mathcal{E}$*

$$\rho(f \circ g) = \rho(f) \circ \mathsf{s}.$$

Proof. Since $g \neq \mathrm{id}_\omega$, there exists $k_g = \min\{k \in \omega \mid k < g(k)\}$. Define $G : \omega \to \omega$ by $G(n) = g^n(k_g)$, where $g^0 = \mathrm{id}_\omega$ and $g^{n+1} = g \circ g^n$. Clearly $G \in \mathcal{E}$. We let $\rho(f) = f \circ G$ for every $f \in \mathcal{E}$. The map $\rho : \mathcal{E} \to \mathcal{E}$ is continuous and for every $f \in \mathcal{E}$ and every n we have

$$\rho(f \circ g)(n) = f \circ g \circ g^n(k_g) = f \circ G(n+1) = (\rho(f) \circ \mathsf{s})(n). \qquad \square$$

Lemma 2.73. *Let $g \in \mathcal{E} \setminus \{\mathrm{id}_\omega\}$. Then $(\mathcal{E}, \vec{\mathsf{s}}) \leqslant_{ch} (\mathcal{E}, \vec{g})$, i.e. there exists a continuous map $\sigma : \mathcal{E} \to \mathcal{E}$ such that for every $f \in \mathcal{E}$*

$$\sigma(f \circ \mathsf{s}) = \sigma(f) \circ g.$$

Proof. Let $k_g = \min\{k \mid k < g(k)\}$. As in the proof of the previous Lemma we define $G \in \mathcal{E}$ by $G(n) = g^n(k_g)$. For every $f \in \mathcal{E}$ and every $l \in \omega$, we let

$$\sigma(f)(l) = \begin{cases} l & \text{if } l < G(0), \\ g^{f(n)-n}(l) & \text{if } G(n) \leqslant l < G(n+1), \text{ for } n \in \omega. \end{cases}$$

Let us check that $\sigma(f)$ is indeed an increasing injection from ω to ω for every $f \in \mathcal{E}$. Since $\sigma(f)$ is increasing and injective on each piece of its definition, it is enough to make the two following observations. Firstly, if $l < G(0)$, then

$$\sigma(f)(l) = l < G(0) \leqslant G \circ f(0) = g^{f(0)}(G(0)) = \sigma(f)(G(0)).$$

Secondly, if $G(n) \leqslant l < G(n+1)$ then

$$\begin{aligned} \sigma(f)(l) = g^{f(n)-n}(l) &< g^{f(n)-n}(G(n+1)) \\ &= g^{f(n)+1}(k_g) \leqslant g^{f(n+1)}(k_g) = G(f(n+1)), \end{aligned}$$

but we have

$$G(f(n+1)) = g^{f(n+1)-(n+1)}(G(n+1)) = \sigma(f)(G(n+1)).$$

One can easily check that $\sigma : \mathcal{E} \to \mathcal{E}$ is continuous. Now on the one hand

$$\sigma(f \circ \mathsf{s})(l) = \begin{cases} l & \text{if } l < G(0), \\ g^{f(n+1)-n}(l) & \text{if } G(n) \leqslant l < G(n+1), \text{ for } n \in \omega. \end{cases}$$

and on the other hand

$$\sigma(f)(g(l)) = \begin{cases} g(l) & \text{if } g(l) < G(0), \\ g^{f(n)-n}(g(l)) & \text{if } G(n) \leqslant g(l) < G(n+1). \end{cases}$$

By definition of G, we have $g(l) < G(0)$ if and only if $l = g(l)$. Moreover if $G(n) \leqslant l < G(n+1)$ then we have $G(n+1) \leqslant g(l) < G(n+2)$ and so

$$\sigma(f \circ \mathsf{s})(l) = g^{f(n+1)-n}(l) = g^{f(n+1)-(n+1)}(g(l)) = \sigma(f)(g(l)),$$

which proves the Lemma. □

3 Sequences in spaces

This chapter is dedicated to the sequences and super-sequences in metric spaces and topological spaces. The results presented here appear in an article [CP14] published by the author and R. Carroy in *Fundamenta Mathematicae*.

We start by making some simple observations on Cauchy sequences in metric spaces. Recall that a sequence $(x_n)_{n\in\omega}$ into a metric space $(\mathcal{X}, d_{\mathcal{X}})$ is Cauchy if for every $\varepsilon > 0$ there exists $k \in \omega$ such that for every $m, n \geqslant k$ we have $d_{\mathcal{X}}(x_m, x_n) < \varepsilon$.

As we have already done we consider sequences as maps from the front $[\omega]^1$. But we now turn $[\omega]^1$ into a metric space by considering this front as a subset of the Cantor space.

The *Cantor space* is the product space 2^ω where 2 is the discrete two points space. A base of clopen sets of 2^ω is given by the sets of the form

$$N_u = \{x \in 2^\omega \mid u \sqsubset x\} \quad \text{for } u \in 2^{<\omega}.$$

For a point $x \in 2^\omega$ a neighbourhood base is given by the sets $N_{x\restriction_n}$ for $n \in \omega$. The space 2^ω admits the compatible metric given by $d(x, y) = 2^{-n-1}$ if $x \neq y$ and n is the least natural number with $x(n) \neq y(n)$. This metric is complete and 2^ω is a very important example of *Polish space*, i.e. a separable completely metrisable topological space. We refer the reader to the monograph by Kechris [Kec95] for basic results about the Cantor space and Polish spaces in general. The Cantor space is compact and zero-dimensional.

We identify each $\{n\}$ with its characteristic function 0^n10^ω in 2^ω, therefore turning $[\omega]^1$ into a subset of 2^ω. This makes $[\omega]^1$ a metric space. Observe that for every positive natural number k,

$$d(\{m\}, \{n\}) < 2^{-k-1} \quad \longleftrightarrow \quad \text{either } m = n\text{, or } m > k \text{ and } n > k.$$

We recall that a function $f : (\mathcal{X}, d_{\mathcal{X}}) \to (\mathcal{Y}, d_{\mathcal{Y}})$ between metric spaces is *uniformly continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, x' \in \mathcal{X}$ we have $d_{\mathcal{X}}(x, x') < \delta$ implies $d_{\mathcal{Y}}(f(x), f(x')) < \varepsilon$.

Fact 3.1. *A sequence $(x_n)_{n\in\omega}$ in a metric space $(\mathcal{X}, d_{\mathcal{X}})$ is Cauchy if and only if the mapping $[\omega]^1 \to \mathcal{X}$, $\{n\} \mapsto x_n$ is uniformly continuous.*

Proof. Suppose that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $m, n \in \omega$, $d(\{n\}, \{m\}) < \delta$ implies $d_{\mathcal{X}}(x_n, x_m) < \varepsilon$. Then for a positive natural number k with $2^{-k-1} < \delta$ we have $n, m > k$ implies $d_{\mathcal{X}}(x_n, x_m) < \varepsilon$, so (x_n) is Cauchy.

Conversely suppose (x_n) is Cauchy and let $\varepsilon > 0$. There exists $k \in \omega$ such that $m, n > k$ implies $d_{\mathcal{X}}(x_m, x_n) < \varepsilon$. So if we take $\delta = 2^{-k-1}$, then for every $m, n \in \omega$, $d(\{m\}, \{n\}) < \delta$ implies $d_{\mathcal{X}}(x_m, x_n) < \varepsilon$. $\square$

For any metric space $\mathcal{X}$ there exists a unique – up to isometry – complete metric space $\widehat{\mathcal{X}}$ such that $\mathcal{X}$ is a dense subspace of $\widehat{\mathcal{X}}$, called the metric completion of $\mathcal{X}$. The metric completion of $\mathcal{X}$ is characterised by following universal property: every uniformly continuous map $f : \mathcal{X} \to \mathcal{Y}$ into a complete metric space extends to a unique continuous map $\hat{f} : \widehat{\mathcal{X}} \to \mathcal{Y}$. Recall that a metric space $\mathcal{X}$ is *totally bounded* if for every $\varepsilon > 0$ there exists a covering of $\mathcal{X}$ by finitely many open balls of radius ε. A metric space is totally bounded if and only if its metric completion is compact. By the Heine-Cantor theorem ([Kec95, (4.5), p. 19]) any continuous map from a compact metric space into a metric space is uniformly continuous. It follows that a map $f : \mathcal{X} \to \mathcal{Y}$ from a totally bounded metric space $\mathcal{X}$ into a complete metric space $\mathcal{Y}$ is uniformly continuous if and only if it extends to unique continuous map $\hat{f} : \widehat{\mathcal{X}} \to \mathcal{Y}$.

The metric completion of $[\omega]^1$ is simply given by its closure inside 2^ω, namely the set $\{0^n 10^\omega \mid n \in \omega\} \cup \{0^\omega\}$ which we identify with $[\omega]^{\leqslant 1}$. Of course, a sequence into a complete metric space $\mathcal{X}$ is Cauchy if and only if it converges. Or in different words, a sequence $(x_n)_{n \in \omega}$ is Cauchy if and only if $[\omega]^1 \to \mathcal{X}$, $\{n\} \mapsto x_n$, extends to a continuous map $[\omega]^{\leqslant 1} \to \mathcal{X}$.

3.1 Cauchy super-sequences

As we did for the front $[\omega]^1$ we now view every front as a subsets of the Cantor space.

Notation 3.2. For any subset u of ω we let $\chi_u \in 2^\omega$ be the characteristic function of u on ω. So for instance, $\chi_{\{2,4\}} = 001010^\omega$ and $\chi_\emptyset = 0^\omega$. For a finite subset s of ω, the restricted characteristic function of s, denoted by $\tilde{\chi}_s$, is $\chi_s \restriction_{\max s+1}$ if $s \neq \emptyset$ and the empty sequence otherwise. For instance, $\tilde{\chi}_{\{2,4\}} = 00101$.

We embed every subset of $[\omega]^{<\infty}$ into the Cantor space via $s \mapsto \chi_s$. By misuse of language, we sometimes identify a subset of ω with its characteristic function inside the Cantor space. In the same way, families of finite sets of natural numbers are sometimes identified with the corresponding subset of the Cantor space. Every front F is therefore considered a metric space for the restriction of the metric on the Cantor space. With the introductory discussion in mind, we make the following definition.

Definition 3.3. Let $\mathcal{X}$ be a metric space. A super-sequence $f : F \to \mathcal{X}$ is said to be *Cauchy* if it is uniformly continuous.

For a family $F \subseteq [\omega]^{<\infty}$ the *closure of* F, denoted by $\overline{F}$, is the topological closure of the set $\{\chi_s \mid s \in F\}$ inside the Cantor space. Of course, the closure of a family F coincides with the metric completion of F. Therefore every Cauchy super-sequence $f : F \to \mathcal{X}$ into a complete metric space $\mathcal{X}$ extends to a continuous map $\bar{f} : \overline{F} \to \mathcal{X}$.

Examples 3.4. (a) The map $f : [\omega]^2 \to 2^\omega$ defined by

$$f(m, n) = 0^m 1 0^n 1 0^\omega$$

is Cauchy. It extends to the continuous map $\bar{f} : [\omega]^{\leqslant 2} \to 2^\omega$ where $\bar{f}(\{m\}) = 0^m 1 0^\omega$ for every $m \in \omega$ and $f(\emptyset) = 0^\omega$.

(b) The map $f : [\omega]^2 \to 2^\omega$ defined by

$$f(m, n) = \begin{cases} 0^\omega & \text{if } n \text{ is even,} \\ 1^\omega & \text{otherwise,} \end{cases}$$

is not Cauchy.

(c) The map $f : [\omega]^2 \to 2^\omega$ defined by

$$f(m, n) = \begin{cases} 1^\omega & \text{if } m + 1 = n, \\ 0^m 1 0^\omega & \text{otherwise,} \end{cases}$$

is not Cauchy. Notice however that for every m the sequence $f(m, n)$ converges trivially to $0^m 1 0^\omega$ when n tends towards infinity. Moreover the sequence $0^m 1 0^\omega$ converges to 0^ω when m tends towards infinity. However the sequence $f(m, m + 1)$ is constant equal to 1^ω while $(m, m + 1)$ converges to $\emptyset$ in 2^ω. Therefore there is no continuous extension of f to the closure of $[\omega]^2$ and so f is not Cauchy.

For a front F, the closure $\overline{F}$ is closely related to the tree $T(F)$ associated to F (cf. Lemma 2.29). The following result is stated by Todorčević [Tod10].

Proposition 3.5. *For every front F we have*

$$\overline{F} = \{\chi_s \in 2^\omega \mid s \in T(F)\}.$$

Proof. Let F be a front on X.

$\supseteq$: Suppose s is in $T(F) \setminus F$, i.e. $s \sqsubset t$ for some $t \in F$. For every $n > \max s$ there is a $u \in F$ with $u \sqsubset s \cup X/n$. If $u \sqsubseteq s$ then we have $F \ni u \sqsubset t \in F$, a contradiction. Hence $s \sqsubset u$ and we found a $u \in F$ with $\chi_u \in N_{\chi_s\restriction_n}$. Since n was arbitrarily large, it follows that $\chi_s \in \overline{F}$.

$\subseteq$: Conversely suppose that an element x of 2^ω belongs to $\overline{F}$. We first show that x is the characteristic function of a finite subset of X. Since $2^X = \{\chi_E \mid E \in \mathcal{P}(X)\}$ is closed in 2^ω and $F \subseteq 2^X$, necessarily x is the characteristic function of a subset of X. Now suppose towards a contradiction that x is the characteristic function of an infinite set $M \subseteq X$. Then for all finite prefix u of M there exists $s \in F$ such that $\chi_s \in N_{x\restriction_{\max(u)+1}}$, and hence $u \sqsubseteq s$. But then $\{u \mid u \sqsubset M\}$ would be an infinite branch of $T(B)$, contradicting Lemma 2.29.

Hence $x = \chi_s$ for some $s \in [X]^{<\infty}$. It only remains to show that there exists a $t \in F$ with $s \sqsubseteq t$. Pick $n > \max s$. Since χ_s belongs to the closure of F there is a $t \in F$ with $\chi_s\restriction_n \sqsubset \chi_t$, and so $s \sqsubseteq t$. □

Remark 3.6. The closure $\overline{F}$ of a front F, being a countable compact metrisable space, admits a natural rank, namely the Cantor-Bendixson rank, denoted by $|\overline{F}|^*_{\mathrm{CB}}$ (see [Kec95, 6.C, p.33]). For every $X \in [\omega]^\infty$ the closure of the front $[X]^k$ has Cantor-Bendixson rank k. Moreover the Schreier barrier $\mathcal{S}$ has Cantor-Bendixson rank ω.

From these examples one could think that for every front F the Cantor-Bendixson rank of $\overline{F}$ is equal to the rank of F. Contrary to what is said by Todorčević [Tod10, Definition 1.24, p.3], this is not the case in general as the following example shows.

Let $F(0) = \{s \in [\omega/0]^{<\infty} \mid \min s = |s|\}$, and $F(n) = [\omega/n]^n$ for $n \geqslant 1$. Then we build the front $F = \mathrm{seq}_{n\in\omega} F(n)$ (cf. Figure 3.1). In fact F is even a *barrier*, namely whenever $s \subseteq t$ in F then $s = t$. Now we have $\mathrm{rk}(F) = \omega + 1$. However the ω^{th} Cantor-Bendixson derivative of $\overline{F}$ is equal to $\{\emptyset, \{0\}\}$, and so we have $|\overline{F}|^*_{\mathrm{CB}} = \omega$. And notably the Cantor-Bendixson rank of $\overline{F_0}$ is also ω.

It also follows from this example that when proving a result about fronts by induction on the Cantor-Bendixson rank of the closure we cannot in general apply the induction hypothesis to the rays. This is the main reason why we prefer $\mathrm{rk}\, F$ to the Cantor-Bendixson rank of $\overline{F}$.

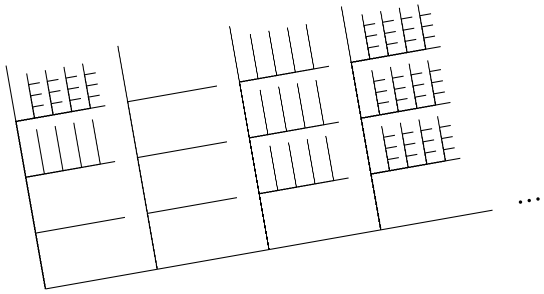


Figure 3.1: A front F with $\operatorname{rk} F = \omega + 1$ and $|\overline{F}|^*_{\mathrm{CB}} = \omega$.

We also note that the closure operation behaves nicely with respect to the taking of restrictions and rays.

Corollary 3.7. *Let F be a front on X.*

1. *For all $M \in [X]^\infty$ we have $\overline{F|M} = \overline{F}|M$;*
2. *For all $n \in X$ we have $\overline{F_n} = \left(\overline{F}\right)_n$.*

Proof. (1) It is enough to prove that $\overline{F}|M \subseteq \overline{F|M}$ holds. So let $s \sqsubset t \in F$ with s a subset of M. Since F is a front there is a $u \in F$ with $u \sqsubset s \cup M/s$ and necessarily $u \in F|M$. If $u \sqsubseteq s \sqsubset t$ we have a contradiction. Hence $s \sqsubseteq u \in F|M$ and so $s \in \overline{F|M}$ by Proposition 3.5.

(2) Since F_n is a front on ω/s, $\overline{F_n} = \{\chi_t \mid \exists u \in F_n\ t \sqsubseteq u\}$ by Proposition 3.5. Now if $t \sqsubseteq u \in F_n$ then $\{n\} \cup t \sqsubseteq \{n\} \cup u \in F$ and thus $t \in \left(\overline{F}\right)_n$. Conversely if $t \in \left(\overline{F}\right)_n$ then $\{n\} \cup t \in \overline{F}$ and thus there exists $u \in F$ with $\{n\} \cup t \sqsubseteq u \in F$. Now $t \sqsubseteq {}_*u \in F_n$ and therefore $t \in \overline{F_n}$. □

Of course a metric space is compact if and only if every sequence admits a converging subsequence. Similarly, a metric space is totally bounded if and only if every sequence admits a Cauchy subsequence. The main result of this chapter (cf. Section 3.3) is the following:

Theorem 3.8. *Let $\mathcal{X}$ be a compact metric space. Then every super-sequence in $\mathcal{X}$ admits a Cauchy sub-super-sequence.*

Using the fact that every non empty compact metric space $\mathcal{X}$ is a continuous image of 2^ω, it will be enough to prove the theorem in the zero-dimensional case. Namely we will prove the following (cf. Theorem 3.25).

Theorem 3.9 (with R. Carroy). *Every super-sequence into 2^ω admits a Cauchy sub-super-sequence.*

The reason for focusing in the zero-dimensional case stems from the existence of a nice and simple characterisation of uniform continuity. Let us observe this already in the simple case of a sequence.

First we recall the following easy characterisation of the clopen sets of 2^ω.

Fact 3.10. *A subset C of 2^ω is clopen if and only if there exists a finite set $A \subseteq 2^{<\omega}$ such that*

(i) *$s, t \in A$ implies $s \not\sqsubseteq t$,*

(ii) *$C = \bigcup_{s\in A} N_s$.*

Proof. Since each N_s is clopen, if $C = \bigcup_{s\in A} N_s$ for a finite set A then C is clopen. Conversely, if C is clopen in 2^ω then $C = \bigcup_{s\in B} N_s$ for some set $B \subseteq 2^\omega$ since the N_s form a base and C is open. Now as C is also closed in the compact space 2^ω it is compact and so there exists a finite set $B' \subseteq B$ such that $C = \bigcup_{s\in B'} N_s$. We can therefore take A to be the $\sqsubseteq$-minimal elements of B'. $\square$

Now we can express uniform continuity of a sequence into a compact metric space in a simple way, reminiscent of topological continuity.

Fact 3.11. *Let $(x_n)_{n\in\omega}$ be a sequence in 2^ω. The following are equivalent:*

(i) *$(x_n)_{n\in\omega}$ is Cauchy;*

(ii) *the map $f : [\omega]^1 \to 2^\omega$, $n \mapsto x_n$ is uniformly continuous;*

(iii) *$f^{-1}(C) = \{n \in \omega \mid x_n \in C\}$ is finite or cofinite for all clopen C of 2^ω.*

Proof. Suppose $(x_n)_{n\in\omega}$ is Cauchy and fix a clopen C of 2^ω. We can write $C = \bigcup_{s\in A} N_s$ for a finite $\sqsubseteq$-antichain $A \subseteq 2^{<\omega}$, and we let $k = \max\{|s| \mid s \in A\}$. There exists $l \in \omega$ such that for all $m, n \geqslant l$ we have $d(x_m, x_n) < 2^{-k-1}$. By definition of the distance this means that $x_m{\restriction}_k = x_n{\restriction}_k$ for all $m, n \geqslant l$. Depending on whether $x_l{\restriction}_k \in A$ or not, we get that $\{n \in \omega \mid x_n \in C\}$ is either cofinite or finite.

Conversely, suppose that $f : [\omega]^1 \to 2^\omega$ is such that $\{n \in \omega \mid x_n \in C\}$ is finite or cofinite for all clopen C of 2^ω. Let $k \in \omega$ and consider the partition $[\omega]^1 = \bigcup_{s\in 2^k} f^{-1}(N_s)$ into preimage of clopen sets. There exists a unique $s \in 2^k$ such that $f^{-1}(N_s)$ is cofinite. Hence there exists $l \in \omega$ such that for all $n \geqslant l$ we have $s \sqsubseteq x_n$ and therefore for all $m, n \geqslant l$ we have $d(x_m, x_n) < 2^{-k-1}$. $\square$

Even though we defined at first Cauchy super-sequences in the setting of metric spaces, this notion is really pertaining to the 'uniform structure' of a front. Our next task is therefore to study the uniform structure of a subset of some compact zero-dimensional space.

3.2 Uniform subspaces of Boolean spaces

Les espaces métriques sont des « espaces uniformes » de nature particulière; les espaces uniformes n'ont été définis de manière générale qu'en 1937, par A. Weil (XI). Auparavant on ne savait utiliser les notions et les résultats relatifs à « structure uniforme » que lorsqu'il s'agissait d'espaces métriques: ce qui explique le rôle important joué dans beaucoup de travaux sur la topologie, par les espaces métriques ou métrisables (et en particulier par les espaces compact métrisables) dans des questions où la distance n'est d'aucune utilité véritable.

Nicolas Bourbaki [Bou06, TG II.43]

The basic reference on uniform spaces is due to Bourbaki [Bou06] and we follow their terminology. A recent account on this topic is provided by Page [Pag11].

Every metric space $(\mathcal{X}, d)$ has an associated uniform structure generated by the entourages of the form $\{(x, y) \in \mathcal{X} \times \mathcal{X} \mid d(x, y) < \varepsilon\}$ for some $\varepsilon > 0$. Very importantly, any compact Hausdorff topological space admits a unique uniform structure that agrees with its topology [Bou06, Theorem 1, II.27]. In particular, every Boolean space, i.e. compact Hausdorff zero-dimensional space, is unambiguously seen as a uniform space.

In this section, we focus on those uniform spaces which arise as a uniform subspace of some Boolean space. We prove a simple characterisation of uniform continuity between such uniform spaces, which is reminiscent of topological continuity. This characterisation is of crucial importance in the next section.

The following notion simplifies greatly the study of these uniform subspaces.

Definition 3.12. Let S be a subset of a Boolean space $\mathcal{X}$. A subset B of S is called a *block* of S (relatively to $\mathcal{X}$) if there exists a clopen C of $\mathcal{X}$ such that $B = C \cap S$. We write $\mathrm{Blocks}(S)$ for the Boolean subalgebra of $\mathcal{P}(S)$ of blocks of S.

The uniform structure of a uniform subspace of a Boolean space $\mathcal{X}$ is essentially given by its blocks as we show in Lemma 3.17 below (see also [Bou06, Exercice 12, II.38]). As a consequence, uniform continuity between such spaces admits of the following simple characterisation.

Proposition 3.13. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Boolean spaces, and let $S \subseteq \mathcal{X}$ and $T \subseteq \mathcal{Y}$ be endowed with the induced uniform structure. Then a function $f : S \to T$ is uniformly continuous if and only if for all $B \in \mathrm{Blocks}(T)$ we have $f^{-1}(B) \in \mathrm{Blocks}(S)$.*

When a subset S of a Boolean space $\mathcal{X}$ is endowed with the induced uniform structure, then the completion of the uniform space S coincide with the closure $\overline{S}$ of S in $\mathcal{X}$[1]. Therefore a function $f : S \to T$ as in Proposition 3.13 is uniformly continuous if and only if there exists a unique continuous map $\overline{f} : \overline{S} \to \overline{T}$ such that $\overline{f}\restriction_S = f$.

Although Proposition 3.13 is folklore, we did not find any reference for this very statement (see however the work of Gehrke, Grigorieff, and Pin [GGP10] in relation with automata theory and the one by Erné [Ern01]). To keep our exposition self-contained we now provide the reader with a series of lemmas that lead to a proof of this fact.

Lemma 3.14. *Let $\mathcal{X}$ be a Boolean space. The unique compatible uniform structure on $\mathcal{X}$ admits*

(i) *as a base the entourages of the form $U_{(C_i)} = \bigcup_i C_i \times C_i$ where (C_i) is a finite partition of $\mathcal{X}$ into clopen sets;*

(ii) *as a subbase the entourages of the form $U_C = (C \times C) \cup (\mathcal{X} \setminus C \times \mathcal{X} \setminus C)$ where C is a clopen set of $\mathcal{X}$.*

Proof. Since $\mathcal{X}$ is compact and Hausdorff, the unique uniform structure which is compatible with the topology of $\mathcal{X}$ consists of the neighbourhoods of the diagonal of $\mathcal{X}$ ([Bou06, Theorem 1, II.27]). Clearly each $U_{(C_i)}$ with (C_i) a clopen partition of $\mathcal{X}$ is a clopen neighbourhood of the diagonal. Conversely let U be a neighbourhood of the diagonal of $\mathcal{X}$. Since $\mathcal{X}$ is zero-dimensional, there exists for each $x \in \mathcal{X}$ a clopen C_x such that $x \in C_x$ and $C_x \times C_x \subseteq U$. By compactness of $\mathcal{X}$, there exist $x_1, \dots, x_n$ such that $C_{x_1}, \dots, C_{x_n}$ covers $\mathcal{X}$. We can then construct a partition $C_1, \dots, C_n$ of $\mathcal{X}$ in clopen such that $C_i \subseteq C_{x_i}$. It follows that $U_{(C_i)} \subseteq U$ which concludes the proof. □

Lemma 3.15. *Let $\mathcal{X}$ be a Boolean space and let F be a closed subspace of $\mathcal{X}$. Then the clopen sets of F coincide with the blocks of F.*

[1] The space $\overline{S}$ is also the Stone dual of the Boolean algebra $\mathrm{Blocks}(S)$, cf. Subsection 4.1.3.

Proof. By definition, if B is a block of F then B is clopen in F. Conversely, let C be clopen in F. The blocks of F form a clopen base of F, and since C is open, it is a union of a family $\mathcal{B}$ of blocks of F. Since F is compact and C closed in F, there is a finite subset $\mathcal{B}'$ of $\mathcal{B}$ for which $C = \bigcup \mathcal{B}'$. Since the blocks of F form a Boolean algebra, it follows that C is a block. □

Lemma 3.16. *Let $\mathcal{X}$ be a Boolean space, and let $S \subseteq T \subseteq \mathcal{X}$. Then*

$$\mathrm{Blocks}(S) = \{B \cap S \mid B \in \mathrm{Blocks}(T)\}.$$

Proof. Let $B \in \mathrm{Blocks}(T)$ and let C be clopen in $\mathcal{X}$ such that $B = C \cap T$. Then $B \cap S = C \cap S$ is a block of S. Conversely, if B is a block of S then there is C clopen in $\mathcal{X}$ with $B = C \cap S$. It follows that $B = (C \cap T) \cap S$ where $C \cap T \in \mathrm{Blocks}(T)$. □

Lemma 3.17. *Let $\mathcal{X}$ be a Boolean space and let S be a subset of $\mathcal{X}$. Then the uniform structure induced on S by $\mathcal{X}$ admits*

(i) *as a base the entourages of the form $U_{(C_i)} = \bigcup_i C_i \times C_i$ where (C_i) is a finite partition of S into blocks;*

(ii) *as a subbase the entourages of the form $U_C = (C \times C) \cup (S \setminus C \times S \setminus C)$ where C is a block of S.*

Proof. Applying Lemmas 3.14 and 3.15 we obtain that the uniform subspace $\overline{S}$ admits as a subbase the entourages of the form U_C where C is a block of $\overline{S}$. Moreover, the uniform subspace S of $\mathcal{X}$ is identical to the uniform subspace S of $\overline{S}$. It follows that entourages of the form $(S \times S) \cap U_C$ where C is a block of $\overline{S}$ constitute a subbase for the uniform space S. On the one hand $(S \times S) \cap U_C = \{(s,t) \in S \times S \mid s \in C \leftrightarrow t \in C\} = U_{C \cap S}$. On the other hand, $\{C \cap S \mid C \in \mathrm{Blocks}(\overline{S})\} = \mathrm{Blocks}(S)$ by Lemma 3.16. We have thus obtained that the entourages of the form U_B where $B \in \mathrm{Blocks}(S)$ constitute a subbase for the uniform space S. □

Proof of Proposition 3.13. $\Rightarrow$: Suppose f is uniformly continuous and let $\hat{f} : \overline{S} \to \overline{T}$ be its continuous extension. Then for all clopen C of $\mathcal{Y}$ the set $f^{-1}(C \cap T) = \hat{f}^{-1}(C) \cap S$ is a block of S.

$\Leftarrow$: Suppose $f : S \to T$ preserves blocks by preimage. By Lemma 3.17, it is enough to show that for each block B of T the preimage of U_B by $f \times f$ is an entourage of S. In fact, $(f \times f)^{-1}(U_B) = U_{f^{-1}(B)}$. □

3.3 Cauchy super-sequences

We endow every front F with the uniform structure inherited from the Cantor space and $\mathrm{Blocks}(F)$ denotes the Boolean algebra of blocks of F:

$$\mathrm{Blocks}(F) = \{C \cap F \mid C \text{ is a clopen of } 2^\omega\}.$$

For an arbitrary family $H \subseteq [\omega]^{<\infty}$ and $n \in \omega$, we recall that the ray of H at n is the family

$$H_n = \{u \in [\omega/n]^{<\infty} \mid \{n\} \cup u \in H\}$$

Examples 3.18. (a) For the front $[\omega]^1$, the Boolean algebra $\mathrm{Blocks}([\omega]^1)$ consists of the finite or cofinite subsets of $[\omega]^1$ (cf. Fact 3.11).

(b) The Boolean algebra of blocks of the front $[\omega]^2$ consists of the subsets B of $[\omega]^2$ such that for every $n \in \omega$ the ray B_n is either finite or cofinite and eitherB_n is empty for cofinitely many n, or $B_n = [\omega/n]^1$ for cofinitely any n.

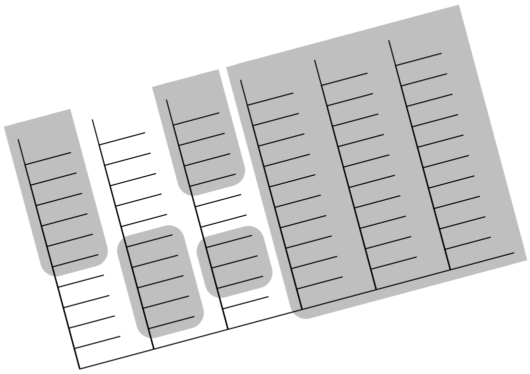


Figure 3.2: A block of the front $[\omega]^2$.

The ray of a family $H \subseteq [\omega]^{<\infty}$ at some $n \in \omega$ is closely related to the following subfamily of H

$$H_{n\sqsubseteq} = \big\{u \in H \mid \{n\} \sqsubseteq u\big\} = \{\chi_u \in H \mid \tilde{\chi}_{\{n\}} \sqsubset \chi_u\} = H \cap N_{\tilde{\chi}_{\{n\}}}.$$

Lemma 3.19. *Let F be a front on N, $n \in N$ and $B \subseteq F$. Then $B_{n\sqsubseteq} \in$ $\mathrm{Blocks}(F)$ if and only if $B_n \in \mathrm{Blocks}(F_n)$.*

Proof. Notice first that $B_{n\sqsubseteq} = B \cap N_{0^n 1}$ and that $F_n \subseteq N_{0^{n+1}}$. Let $\eta_n : N_{0^{n+1}} \to N_{0^n 1}$ be the homeomorphism given by $\eta_n(S) = \{n\} \cup S$ for every $S \in \mathcal{P}(\omega/n) = N_{0^{n+1}}$. Suppose that $B_{n\sqsubseteq} \in \mathrm{Blocks}(F)$ and let $C \subseteq N_{0^n 1}$ be clopen in 2^ω with $B_{n\sqsubseteq} = F \cap C$. Then $\eta_n^{-1}(C)$ is clopen and $B_n = \eta_n^{-1}(C) \cap F_n$, so B_n is a block of F_n. Conversely suppose that $B_n \in \mathrm{Blocks}(F_n)$ and let $C \subseteq N_{0^{n+1}}$ be clopen in 2^ω with $B_n = F_n \cap C$. Then $B_{n\sqsubseteq} = F \cap \eta_n(C)$, and as $\eta_n(C)$ is clopen we have $B_{n\sqsubseteq} \in \mathrm{Blocks}(F)$. □

The following proposition gives a useful characterisation of the family of blocks of a front.

Proposition 3.20. *We have the following.*

(i) *If F is the trivial front, then* $\mathrm{Blocks}(F) = \mathcal{P}(F)$.

(ii) *If F is a non-trivial front and B is a subset of F, then $B \in \mathrm{Blocks}(F)$ if and only if*

  (a) *B_n is a block of F_n for every $n \in N$, and*

  (b) *either for all but finitely many $n \in N$ we have $B_n = \emptyset$, or for all but finitely many $n \in N$ we have $B_n = F_n$.*

Proof. The statement (i) is trivial. For (ii), suppose F is a non-trivial front and assume that $B \subseteq F$ satisfies Conditions (a) and (b). For every $n \in N$ the set B_m is a block of F_m and so by Lemma 3.19 $B_{m\sqsubseteq}$ is a block of F. Notice that B is equal to the disjoint union of the $B_{n\sqsubseteq}$ for $n \in N$. If B_n is empty for all but finitely many n, then B is a block as a finite union of blocks. Now if there exists k such that $B_n = F_n$ for every $n \in N/k$, then

$$B = \bigcup_{\substack{m \in N \\ m \leqslant k}} B_{m\sqsubseteq} \cup F|(N/k)$$

is also a block as a finite union of blocks, since $F|(N/k) = F \cap N_{0^{k+1}}$.

Conversely, if B is a block of F then there exists a clopen C of 2^ω with $B = F \cap C$. In particular $B_{n\sqsubseteq} = F \cap C \cap N_{0^n 1}$ is a block of F for every $n \in N$, and so B_n is a block of F_n by Lemma 3.19. Now we can write the clopen C as $\bigcup_{s \in A} N_s$ for a finite $\sqsubseteq$-antichain $A \subseteq 2^{<\omega}$. Let $A_n = \{s \in A \mid 0^n 1 \sqsubseteq s\}$ for every $n \in \omega$ and notice that $B_{n\sqsubseteq} = F \cap \bigcup_{s \in A_n} N_s$. Then either $A = \bigcup_{n \in \omega} A_n$ or there exists l such that $A = \{0^l\} \cup \bigcup_{n \in \omega} A_n$. In the first case, it follows that B_n is empty for all but finitely many $n \in N$. In the second case, it follows that $B_n = F_n$ for every $n > l$. In both cases the set B fulfils Conditions (a) and (b). □

Remark 3.21. The previous Proposition actually provides us with a definition of $\text{Blocks}(F)$ for every front F by induction on the rank.

Next we notice that the notion of block behaves nicely with the taking of restrictions. Observe that for $S \subseteq T \subseteq [\omega]^{<\infty}$ and $N \in [\omega]^\infty$ we have

$$S|N = S \cap [N]^{<\infty} = S \cap 2^N = S \cap T \cap 2^N = S \cap T|N,$$

hence the restriction of S to N equals the trace of S on $T|N$.

Notation 3.22. Let $F \subseteq [\omega]^{<\infty}$. For a family $\mathbf{S}$ of subsets of F and $N \in [\omega]^\infty$ we denote by $\mathbf{S}|N$ the family

$$\{S|N \mid S \in \mathbf{S}\} = \{S \cap F|N \mid S \in \mathbf{S}\}.$$

In fact, the blocks of the restriction of a front are simply the restrictions of the blocks of the front.

Lemma 3.23. *Let F be a front on M and let $N \subseteq M$ be infinite. Then*

$$\text{Blocks}(F)|N = \text{Blocks}(F|N).$$

Proof. If $B \in \text{Blocks}(F)$ then there exists C clopen in 2^ω with $B = C \cap F$. It follows that

$$B|N = B \cap F|N = B \cap F \cap F|N = C \cap F|N$$

is a block of $F|N$.

Conversely, if we have $B = C \cap F|N$ for some clopen C of 2^ω then for the block $B' = C \cap F$ of F we have $B'|N = B' \cap F|N = C \cap F \cap F|N = B$. □

We observe that for a finite family $\mathbf{F}$ of subsets of a front F on M we can find by repeated application of Nash-Williams' Theorem 2.36 an infinite $N \subseteq M$ such that for all $S \in \mathbf{F}$, $S|N = S \cap F|N$ is either empty or equal to $F|N$. In other terms, for $\mathbf{F}$ a finite family of subsets of a front F on M, there exists $N \in [M]^\infty$ such that $\mathbf{F}|N \subseteq \{\emptyset, F|N\}$.

Of course such a conclusion cannot be expected for a countably infinite family $\mathbf{F}$, as one easily see by considering for example the countably infinite family $\mathbf{F} = \big\{\{s\} \mid s \in F\big\}$ of subsets of a non-trivial front F. However we obtained the following which is the main combinatorial result in this chapter.

Theorem 3.24 (with R. Carroy)**.** *Let F be a front on ω and let $\mathbf{S}$ be a countable family of subsets of F. For all $M \in [\omega]^\infty$ there exists $N \in [M]^\infty$ such that $\mathbf{S}|N$ consists of blocks of $F|N$, i.e. $\mathbf{S}|N \subseteq \text{Blocks}(F|N)$.*

Proof. By induction on the rank of F. For the trivial front, the theorem is trivial.

Suppose that F is a front of strictly positive rank on ω and that the statement of the theorem holds for fronts of strictly smaller rank. Let $M \in [\omega]^\infty$.

Claim. *There exists $X \in [M]^\infty$ such that for all $m \in X$ and all $S \in \mathbf{S}$ we have $(S|X)_m \in \operatorname{Blocks}\left((F|X)_m\right)$.*

Proof of the claim. For each $n \in \omega$, since F_n is a front of strictly smaller rank than F, we can apply our induction hypothesis to the countable family $\mathbf{S}_n = \{S_n \mid S \in \mathbf{S}\}$ of subsets of F_n. We thus build recursively a sequence $(X_i)_{i\in\omega}$ of infinite subsets of M with $k_i = \min(X_i)$, such that

(i) $X_0 = M$ and $X_{i+1} \in [X_i/k_i]^\infty$ for every $i \in \omega$;

(ii) for every $i \in \omega$ we have $\mathbf{S}_{k_i}|X_{i+1} \subseteq \operatorname{Blocks}\left(F_{k_i}|X_{i+1}\right)$.

By induction, suppose that the sequence is defined up to $i \geqslant 0$ and let $k_i = \min X_i$. Then F_{k_i} is a front on M/k_i whose rank is strictly smaller than the rank of F. We consider the family $\mathbf{S}_{k_i}$ of subsets of F_{k_i}. By our induction hypothesis there exists $X_{i+1} \in [X_i/k_i]^\infty$ such that $\mathbf{S}_{k_i}|X_{i+1} \subseteq \operatorname{Blocks}\left(F_{k_i}|X_{i+1}\right)$.

We then set $X = \{k_i \mid i \in \omega\}$. To see that X satisfies the Claim, let $k_i \in X$ and $S \in \mathbf{S}$. Since $X/k_i \subseteq X_{i+1}$ we have

$$(S|X)_{k_i} = S_{k_i}|X = S_{k_i}|X/k_i = S_{k_i}|X_{i+1}|X/k_i \in \operatorname{Blocks}\left(F_{k_i}|X_{i+1}\right)|X/k_i,$$

by the choice of X. Therefore by Lemma 3.23, it follows that

$$(S|X)_{k_i} \in \operatorname{Blocks}\left(F_{k_i}|X/k_i\right) = \operatorname{Blocks}\left((F|X)_{k_i}\right),$$

as desired. □

By the Claim, there is no loss of generality in assuming that F is a front on M and that for all $m \in M$ and all $S \in \mathbf{S}$ we have $S_m \in \operatorname{Blocks}(F_m)$.

We now fix an enumeration $\{S^i \mid i \in \omega\}$ of $\mathbf{S}$. We build a sequence $(N_i)_{i\in\omega}$ of infinite subsets of M with $n_i = \min N_i$ such that

(i) $N_0 = M$ and $N_{i+1} \in [N_i/n_i]^\infty$;

(ii) for every $i \in \omega$ and every $j < i$, either $S^j|N_i$ is empty, or $S^j|N_i = F|N_i$.

By induction suppose that the sequence is defined up to $i \geqslant 0$. Applying Nash-Williams' Theorem $i+1$ many times, there exists $N_{i+1} \in [N_i/n_i]^\infty$ such that for every $j < i+1$ either $S^j|N_{i+1}$ is empty or equal to $F|N_{i+1}$.

Now for $N = \{n_0, n_1, ...\}$, we claim that for all $S \in \mathbf{S}$ we have $S|N$ is a block of $F|N$. To see this let $S^j \in \mathbf{S}$. Firstly, for every i we have

$$(S^j|N)_{n_i} = S^j_{n_i}|N \in \text{Blocks}\,(F_{n_i})|N = \text{Blocks}\,(F|N)_{n_i}),$$

by Lemma 3.23. Secondly, since $N/k_j \subseteq N_{j+1}$ we have

$$S^j|(N/k_j) = (S^j|N_{j+1})|N/k_j$$

and $S^j|N_{j+1}$ is either empty or equal to $F|N_{j+1}$ by the choice of N. It follows that either $(S^j|N)_m$ is empty for every $m \in N/k_j$ or $(S^j|N)_m = (F|N)_m$ for every $m \in N/k_j$. We therefore conclude by Proposition 3.20 that $S^j|N$ is a block of $F|N$ for every $j \in \omega$ as desired. □

We can now come to the main result of this section.

Theorem 3.25 (with R. Carroy)**.** *Let F be a front on some $M \in [\omega]^\infty$. For all $f : F \to 2^\omega$ there exists an infinite $N \subseteq M$ such that the restriction $f : F|N \to 2^\omega$ is uniformly continuous.*

Proof. Applying Theorem 3.24 to $\mathbf{S} = \{f^{-1}(C) \mid C \text{ is clopen in } 2^\omega\}$ we obtain an infinite $N \subseteq \omega$ such that the restriction $f|N : F|N \to X$ satisfies that for all clopen set C we have

$$(f|N)^{-1}(C) = f^{-1}(C) \cap F|N = f^{-1}(C)|N \in \text{Blocks}(F|N).$$

Therefore $f|N$ is uniformly continuous by Proposition 3.13. □

This result generalises to arbitrary compact metric space using the fact that every such space is a continuous image of the Cantor space as we already noticed in Theorem 3.8.

Theorem 3.26. *Let F be a front on some $M \in [\omega]^\infty$ and $\mathcal{X}$ be a compact metric space. For all $f : B \to \mathcal{X}$ there exists an infinite $N \subseteq M$ such that the restriction $f : B|N \to \mathcal{X}$ is uniformly continuous.*

Proof. If $\mathcal{X}$ is a non-empty compact metric space, then there exists a continuous $h : 2^\omega \to \mathcal{X}$ onto $\mathcal{X}$ [Kec95, (4.18), p. 23], and this h is in fact uniformly continuous by the Heine-Cantor theorem [Kec95, (4.5), p. 19]. So let $f : F \to \mathcal{X}$ be any super-sequence. We can choose some super-sequence

$g : F \to 2^\omega$ such that $h \circ g = f$. Now applying Theorem 3.25 to g, there exists some front $F' \subseteq F$ such that $g{\restriction}_{F'} : F' \to 2^\omega$ is uniformly continuous. It follows that $f{\restriction}_{F'} = h \circ g{\restriction}_{F'} : F' \to \mathcal{X}$ is uniformly continuous as a composition of uniformly continuous maps. □

In Subsection 2.2.3 we considered $[X]^\infty$ for $X \in [\omega]^\infty$ with the induced topology from 2^ω. We noticed it was homeomorphic to the Baire space and we viewed it as the closed subset of ω^ω consisting of strictly increasing sequences in X.

However as a subspace of 2^ω, the set $[X]^\infty$ also receives a unique uniformity. As a uniform space $[X]^\infty$ is not complete, its completion is of course the Cantor space 2^X. The Boolean algebra of blocks of $[X]^\infty$ consists of finite union of sets of the form

$$N(F, G) = \{Y \in [X]^\infty \mid F \subseteq Y \text{ and } G \cap Y = \emptyset\}, \quad \text{for } F, G \in [X]^{<\infty}.$$

We have an analogue to Theorem 3.8 in the context of locally constant maps from $[\omega]^\infty$ into a compact metric space.

Theorem 3.27. *Let $\mathcal{X}$ be compact metrisable and $h : [\omega]^\infty \to \mathcal{X}$ be locally constant. There exists $Y \in [\omega]^\infty$ such that the restriction $h : [Y]^\infty \to \mathcal{X}$ is uniformly continuous, and therefore extends to a continuous map $\overline{h} : 2^Y \to \mathcal{X}$.*

Proof. By Theorem 3.26 there exists some $Y \in [\omega]^\infty$ such that the restriction $\check{h} : F^h|Y \to \mathcal{X}$ is uniformly continuous. This Y is as required since $h : [Y]^\infty \to \mathcal{X}$ extends to the continuous $\overline{h} : 2^Y \to \mathcal{X}$ given by $\overline{h}(s) = \overline{\check{h}}(s)$ if $s \in \overline{F^h|Y}$ and $\overline{h}(u) = \check{h}(s)$ for the unique $s \in F^h|Y$ with $s \sqsubset u$, otherwise. □

3.4 Converging super-sequences

One can think of a super-sequence $f : F \to \mathcal{X}$ which extends to a continuous $\overline{f} : \overline{F} \to \mathcal{X}$ as 'converging towards $\overline{f}$'. In this section, we study these 'converging super-sequences', namely the continuous maps $f : \overline{F} \to \mathcal{X}$ into some topological space $\mathcal{X}$.

3.4.1 Normal form

Let $f : \overline{F} \to \mathcal{X}$ be a map from the closure of a front F on some $N \in [\omega]^\infty$ into a topological space $\mathcal{X}$. Let

$$\Lambda_f = \{s \in \overline{F} \mid f(s) \text{ is limit in } \mathcal{X}\},$$

the closed subset of $\overline{F}$ of those points whose image is a limit point in $\mathcal{X}$. Observe that if $M \in [N]^\infty$ and $g = f{\restriction}_{\overline{F|M}} : \overline{F}|M \to \mathcal{X}$, then $\Lambda_g = \Lambda_f|M$. Recall that by Corollary 3.7, for all $M \in [N]^\infty$ we have $\overline{F}|M = \overline{F|M}$ and $\left(\overline{F}\right)_n = \overline{F_n}$ for all $n \in N$.

Definition 3.28. Let F be a front on $N \in [\omega]^\infty$ and $\mathcal{X}$ a topological space. We say that a continuous map $f : \overline{F} \to \mathcal{X}$ is *normal* if

(1) Λ_f is either empty or the closure of a front on X,

(2) for all $s, t \in \overline{F} \setminus \Lambda_f$, $s \sqsubseteq t$ implies $f(s) = f(t)$.

Example 3.29. The map $f : [\omega]^{\leqslant 2} \to [\omega]^{\leqslant 2}$ given by

$$f(m,n) = \begin{cases} 0^\omega & \text{if } m = 0 \text{ and } n = 1, \\ 0^m 1 0^n 1 0^\omega & \text{otherwise,} \end{cases}$$

$f(m) = 0^m 1 0^\omega$ and $f(\emptyset) = 0^\omega$, is continuous but it is not normal since $\Lambda_f = [\omega]^{\leqslant 1} \cup \{(0,1)\}$ is not the closure of a front.

Lemma 3.30. *If $f : \overline{F} \to \mathcal{X}$ is normal and $G \subseteq F$ is a front then $g = f{\restriction}_{\overline{G}} : \overline{G} \to \mathcal{X}$ is normal too.*

Proof. Let $M \in [\bigcup F]^\infty$ with $G = F|M$. Then $\Lambda_g = \Lambda_f|M$ is either empty or a front on M. Moreover if $s, t \in \overline{G} \setminus \Lambda_g$, then $s, t \in \overline{F}$, $s, t \subset M$, and $s, t \notin \Lambda_f|M$, so $s, t \notin \Lambda_f$ and thus $s \sqsubseteq t$ implies $g(s) = f(s) = f(t) = g(t)$. $\square$

Here is Theorem 1.2 from the Introduction.

Theorem 3.31 (with R. Carroy)**.** *Let F be a front on $N \in [\omega]^\infty$, $\mathcal{X}$ be a topological space and $f : \overline{F} \to \mathcal{X}$ be a continuous map. Then there exists a front $G \subseteq F$ such that $g = f{\restriction}_{\overline{G}} : \overline{G} \to \mathcal{X}$ is normal.*

Proof. By induction on the rank of F. The theorem is obvious for the trivial front. We suppose it holds for all continuous maps from the closure of a front with rank strictly smaller than $\alpha > 0$. Let F be a front on N with $\operatorname{rk} F = \alpha$.

Claim. *There exists $M \in [N]^\infty$ such that for all $m \in M$ the map*

$$\begin{aligned} f_m^M : \left(\overline{F|M}\right)_m &\longrightarrow \mathcal{X} \\ s &\longmapsto f(\{m\} \cup s) \end{aligned}$$

is normal.

Proof of the claim. We build by recursion a sequence $(M_i)_{i\in\omega}$, with m_i the minimum of M_i, such that for all $i \in \omega$ we have

(a) $M_0 = N$ and $M_{i+1} \in [M_i/m_i]^\infty$,

(b) $f_{m_i}^{M_i} : \overline{(F|M_{i+1})}_{m_i} \to \mathcal{X}$ is normal.

By induction, suppose that the sequence is defined up to $i \geqslant 0$. The family F_{m_i} is a front on M_i/m_i with $\mathrm{rk}(F_{m_i}) < \alpha$, so we can use the induction hypothesis on the continuous map $f_{m_i} : \overline{F_{m_i}} \to \mathcal{X}$ defined by $s \mapsto f(\{m_i\} \cup s)$ to get $M_{i+1} \in [M_i/m_i]^\infty$ such that the restriction $f_{m_i}^{M_{i+1}} : \overline{(F|M_{i+1})}_{m_i} \to \mathcal{X}$ is normal.

Now setting $M = \{m_0, m_1, ...\}$ we have $M/m_i \in [M_{i+1}]^\infty$ and thus $f_{m_i}^{M/m_i}$ is normal by Lemma 3.30. □

Therefore we can suppose without loss of generality that F is a non-trivial front on N and that $f : \overline{F} \to \mathcal{X}$ is such that

$$\begin{aligned} f_n : \overline{F}_n &\longrightarrow \mathcal{X} \\ s &\longmapsto f(\{n\} \cup s) \end{aligned}$$

is normal for all $n \in N$. Notice that as F is a non-trivial front on N, $\{n\} \in \overline{F}$ for all $n \in N$. We now distinguish two cases.

$\emptyset \notin \Lambda_f$*:* Since f is continuous we have $f(\emptyset) = \lim_{n\in N} f(\{n\})$ and as $f(\emptyset)$ is isolated in $\mathcal{X}$ there exists a $M \in [N]^\infty$ such that $f(\{m\}) = f(\emptyset)$ for all $m \in M$. Then for all $m \in M$ we have that $\{m\} \notin \Lambda_f$, that is $\emptyset \notin (\Lambda_f)_{\{m\}}$. Hence $(\Lambda_f)_{\{m\}}$ cannot be the closure of a front, so it must be empty. Therefore $\Lambda_f|M$ is empty and for $G = F|M$, the restriction $f\restriction_{\overline{G}} : \overline{G} \to \mathcal{X}$ is constant, hence normal.

$\emptyset \in \Lambda_f$*:* There exists $M \in [N]^\infty$ such that either $\{m\} \notin \Lambda_f$ for all $m \in M$, or $\{m\} \in \Lambda_f$ for all $m \in M$.

- In the former case, we have $\Lambda_f|M = \{\emptyset\}$ and so $G = F|M$ meets the conditions.
- In the latter case, for all $m \in M$ the set $(\Lambda_f|M)_m$ is the closure of a front L_m on M/m. Hence $L = \mathrm{seq}_{m\in M} L_m$ is a front on M and $\Lambda_f|M = \overline{L}$. Hence $G = F|M$ meets the requirements. □

Remark 3.32. It does not hold that for every map $f : \overline{F} \to 2$ from the closure of a front there exists a front $G \subseteq F$ such that at least one of $f{\restriction}_{\overline{G}}^{-1}(0)$ or $f{\restriction}_{\overline{G}}^{-1}(1)$ is either empty or the closure of a front. One can for example consider the function $f : [\omega]^{\leqslant 2} \to 2$ defined by

$$f(m,n) = \begin{cases} 0 & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd,} \end{cases} \qquad f(m) = \begin{cases} 1 & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even,} \end{cases} \qquad f(\emptyset) = 0.$$

3.4.2 Super-sequences of isolated points

In Section 4.2 we deal with super-sequences $f : F \to \mathcal{X}$ which range in $\operatorname{Isol}(\mathcal{X})$, the isolated points of $\mathcal{X}$, and which continuously extend to a map $\bar{f} : \overline{F} \to \mathcal{X}$. In this case, the set $\Lambda_{\bar{f}}$ is then included in $\overline{F} \setminus F$. We collect here some easy lemmas pertaining to this situation.

Lemma 3.33. *If F is a front then $\overline{F} \setminus F = \{\chi_s \mid \exists t \in F \; s \sqsubset t\}$ and $F = \max_{\sqsubseteq} \overline{F}$.*

Proof. By Proposition 3.5 we have $\{\chi_s \mid \exists t \in F \; s \sqsubseteq t\} = \overline{F}$. For the first claim, if $s \sqsubseteq t \in F$ and $s \notin F$ then $s \sqsubset t$. Conversely, if $s \sqsubset t \in F$ then $s \notin F$ since F is a front. For the second claim, if $s \in F$ and $s \sqsubseteq t \in \overline{F}$, then there is $u \in F$ with $t \sqsubseteq u$, so $s \sqsubseteq u$ and thus $s = t = u$. And if $s \in \overline{F}$ is maximal in $\overline{F}$ then there is no $t \in F$ with $s \sqsubset t$ and thus by the first claim $s \in F$. □

Lemma 3.34. *If F and G are fronts and $G \subseteq \overline{F} \setminus F$ then $1 + \operatorname{rk} G \leqslant \operatorname{rk} F$.*

Proof. Notice that $T(G) \subseteq T(F)$. By induction we show that for every $s \in T(G)$ we have $1 + \rho_{T(G)}(s) \leqslant \rho_{T(F)}(s)$. If $s \in G$ then there exists $t \in F$ with $s \sqsubset t$ and so $1 + \rho_{T(G)}(s) = 1 \leqslant \sup\{\rho_{T(F)}(t) + 1 \mid s \sqsubset t \in T(F)\} = \rho_{T(F)}(s)$. Now if $s \in T(G) \setminus G$ then by induction hypothesis

$$\begin{aligned} 1 + \rho_{T(G)}(s) &= \sup\{(1 + \rho_{T(G)}(t)) + 1 \mid s \sqsubset t \in T(G)\} \\ &\leqslant \sup\{\rho_{T(F)}(t) + 1 \mid s \sqsubset t \in T(F)\} = \rho_{T(F)}(s). \end{aligned}$$

Therefore $1 + \operatorname{rk} G = 1 + \rho_{T(G)}(\emptyset) \leqslant \rho_{T(F)}(\emptyset) = \operatorname{rk} F$. □

Remark 3.35. The topology induced by the Cantor space on every front F is discrete. It easily follows from this fact that the set of isolated points $\operatorname{Isol}(\overline{F})$ of the closure of F is equal to F, or in other words the Cantor-Bendixson derivative of $\overline{F}$ is equal to F. However, in general it is not true that $\overline{F} \setminus F$ is the closure of a front as the following example shows.

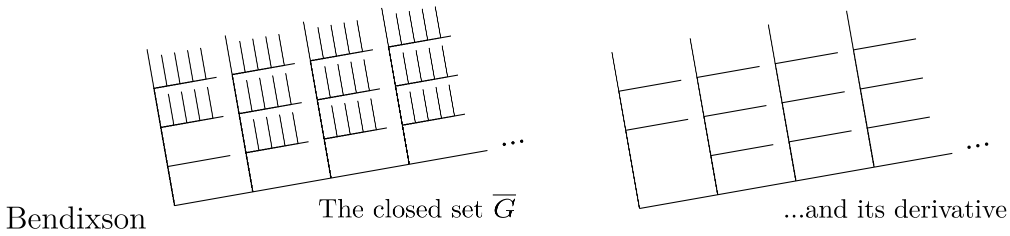


Figure 3.3: A front whose derivative of the closure is no closure of a front.

Let $G(0) = \{\{1\}\} \cup [\omega/1]^2$ and $G(n) = [\omega/n]^2$ for every $n \geqslant 1$. We build the front $G = \operatorname{seq}_{n\in\omega} G(n)$ (cf. Figure 3.3). We have

$$G' = \operatorname{Isol}\left(\overline{G} \setminus G\right) = \left\{\{0, n\} \mid n \geqslant 2\right\} \cup [\omega/1]^2.$$

But while $\bigcup G' = \omega$, the infinite set ω has no initial segment in G'. Hence G' is not a front.

Lemma 3.36. *Let F be a front, $\mathcal{X}$ a topological space. If $f : F \to \operatorname{Isol}(\mathcal{X})$ is spare and extends to a normal $f : \overline{F} \to \mathcal{X}$, then $\Lambda_f = \overline{F} \setminus F$.*

Proof. Since $f : F \to \operatorname{Isol}(\mathcal{X})$ we have $\Lambda_f \subseteq \overline{F} \setminus F$ by definition. Let $s \in \overline{F} \setminus \Lambda_f$. Since $f : \overline{F} \to \mathcal{X}$ is normal then for every $t \in F$ we have $s \sqsubseteq t$ implies $f(s) = f(t)$. Since $f : F \to \mathcal{X}$ is spare, necessarily $s \in F$ by Lemma 2.45. $\square$

In general for a front F, $\overline{F} \setminus F$, which is the Cantor-Bendixson derivative of $\overline{F}$, is not of the form $\overline{G}$ for a front G. However when this happens, as in the previous lemma, we have the following result.

Lemma 3.37. *Let F be a front on X. If $\overline{F} \setminus F = \overline{G}$ for some front G on X, then for every $s \in \overline{G}$ the following are equivalent*

(i) *$s \in G$,*

(ii) *for every $n \in X/s$, $s \cup \{n\} \in F$,*

(iii) *there exists $n \in X/s$ such that $s \cup \{n\} \in F$.*

And moreover,

$$F = \{s \cup \{n\} \mid s \in G \text{ and } n \in X/s\}.$$

Proof. (i)→(ii)*:* Let $n \in X/s$. Since F is a front on X, there exists $t \in F$ with $t \sqsubset s \cup \{n\} \cup X/n$. Now $t \sqsubseteq s$ would contradict Lemma 3.33 for F. And if $s \cup \{n\} \sqsubset t$ then $s \cup \{n\} \in \overline{F} \setminus F = \overline{G}$ by Lemma 3.33 and so $s \sqsubset s \cup \{n\} \in \overline{G}$ contradicting Lemma 3.33 for G. So $s \cup \{n\} = t \in F$ as desired.

(iii)→(i)*:* If for some $n \in X/s$ we have $s \cup \{n\} \in F$, then by Lemma 3.33 s is maximal for $\sqsubseteq$ in $\overline{G}$ and so $s \in G$.

Finally by (ii) $s \cup \{n\} \in F$ for every $s \in G$ and $n \in X/s$. Conversely if $s \cup \{n\} \in F$ then $s \in \overline{G}$ and thus $s \in G$ by (iii). □

This concludes this chapter on super-sequences in metric and topological spaces. The results we obtained are applied in the context of BQO theory in the next chapter.

4 The ideal space of a well-quasi-order

For simplicity we restrict ourselves to partial orders in this chapter. This is of course no real restriction since we are only interested in the properties of quasi-orders preserved by equivalence and every quasi-order admits a unique equivalent partial order up to isomorphism.

The following 'ideal objects' of a partial ordered play a central rôle in this chapter.

Definition 4.1. Let P be a partial order. An *ideal* (also directed ideal, up-ideal) of P is a subset $I \subseteq P$ such that

(1) I is non empty;

(2) I is a downset;

(3) for every $p, q \in I$ there exists $r \in I$ with $p \leqslant r$ and $q \leqslant r$.

Equivalently a subset I of a po P is an ideal if I a downset and I is *directed*, namely every (possibly empty) finite subset $F \subseteq I$ admits an *upper bound* in I, i.e. there exists $q \in I$ with $F \subseteq \downarrow q$. For every $p \in P$, the set $\downarrow p$ is an ideal called a *principal* ideal.

Definition 4.2. The set of ideals of P partially ordered by inclusion, denoted by $\mathrm{Id}(P)$, is referred to as the *ideal completion* of the partial order P.

For each partial order P we have the embedding

$$\begin{aligned} P &\longmapsto \mathrm{Id}(P) \\ p &\longmapsto \downarrow p, \end{aligned}$$

and we henceforth identify each element p with the corresponding principal ideal $\downarrow p$. In particular we have the inclusions $P \subseteq \mathrm{Id}(P) \subseteq \mathcal{D}(P)$ as partial orders.

We observe that we cannot replace $\mathcal{D}(P)$ by $\mathrm{Id}(P)$ in Proposition 2.6, i.e. it is *not* true that a po P is WQO if and only $\mathrm{Id}(P)$ is well-founded. The simplest example is given by the antichain $A = (\omega, =)$. The partial order $\mathrm{Id}(A)$ is equal to A so, in particular, even though A is not WQO, $\mathrm{Id}(A)$ is well-founded. Nonetheless, when P is WQO then $\mathrm{Id}(P)$ is well-founded.

Recall also that by Corollary 2.56 if P is BQO, then $V^*(P)$ is BQO, and so in particular $\mathrm{Id}(P)$ – which is included in $V^*(P)$ – is BQO. Of course since $P \subseteq \mathrm{Id}(P)$, it follows that P is BQO if and only if $\mathrm{Id}(P)$ is BQO.

A *non-principal* ideal of P is an ideal which is not of the form $\downarrow p$ for some $p \in P$. We write $\mathrm{Id}^*(P)$ for the partial order of non-principal ideals, i.e. $\mathrm{Id}^*(P) = \mathrm{Id}(P) \setminus P$. The partial order $\mathrm{Id}^*(P)$ is therefore the remainder of the ideal completion of P.

The main new result in this chapter is a proof of the following conjecture due to Pouzet [Pou78]: If P is WQO and $\mathrm{Id}^*(P)$ is BQO, then P is BQO.

Notice that the assumption that P is WQO cannot be dropped since for both the antichain $(\omega, =)$ and the descending chain ω^{op} – the opposite of $(\omega, \leqslant)$ – the remainder of the ideal completion is empty, hence BQO.

Our proof of Pouzet's conjecture relies on the fact that the ideal completion $\mathrm{Id}(P)$ of a WQO P is actually a compact topological space when topologised as a subspace of the generalised Cantor space 2^P and that the results we obtained in Chapter 3 can eventually be applied to yield the following (cf. Theorem 4.34):

Theorem 4.3. *Every super-sequence $f : F \to P$ into a* WQO *P admits a Cauchy sub-super-sequence $f' : F' \to P$ which therefore extends to a continuous map $\overline{f'} : \overline{F'} \to \mathrm{Id}(P)$ into the ideals of P.*

Using the above theorem and the results of Chapter 3 we then prove (cf. Theorem 4.39) that any bad super-sequence in a WQO P yields a bad super-sequence into the non principal ideals of P. This shows that if a WQO P is not BQO, then $\mathrm{Id}^*(P)$ is not BQO, therefore proving Pouzet's conjecture.

Our first task is to collect the properties of the ideal completion of a WQO which are required to prove the above theorem. We do this in the first section by showing that in the case of a WQO the ideal completion coincides with two other important completions of a quasi-order – namely the Cauchy ideal completion and the profinite completion or Nachbin order-compactification – the properties of which combine to yield what we call the *ideal space* of a WQO.

4.1 Completions of well-quasi-orders

4.1.1 Ideal completion

Let us start by the well-known fact that ideals play a particular rôle among the downsets of a partial order.

Proposition 4.4. *For a* non empty *downset I of a partial order P, the following are equivalent.*

(i) *I is an ideal;*

(ii) *I is a* join-prime *element of the lattice $\mathcal{D}(P)$, i.e. for every $E, D \in \mathcal{D}(P)$, $I \subseteq E \cup D$ implies $I \subseteq E$ or $I \subseteq D$;*

(iii) *for every $E, D \in \mathcal{D}_{fb}(P)$, $I \subseteq E \cup D$ implies $I \subseteq E$ or $I \subseteq D$.*

Proof. (i)→(ii): For an ideal I and downsets D and E, suppose that $I \not\subseteq D$ and $I \not\subseteq E$. Then there exist $p, q \in I$ with $p \notin D$ and $q \notin E$, and since I is directed there is some $r \in I$ with $r \geqslant p$ and $r \geqslant q$. As E and D are downsets, necessarily r does not belong to D nor to E. Therefore $I \not\subseteq E \cup D$.

(ii)→(iii): Trivial.

(iii)→(i): Suppose I is a non empty downset which is not directed. Then there exist $p, q \in I$ such that $\uparrow p \cap \uparrow q \cap I$ is empty. Therefore $I \subseteq P \setminus \uparrow p \cup P \setminus \uparrow q$ while both $I \not\subseteq P \setminus \uparrow p$ and $I \not\subseteq P \setminus \uparrow q$ hold. □

A partial order in which every directed set D admits a supremum, denoted by $\sup D$, is called a *directed complete partial order* (dcpo, also up-complete). Since any directed union of ideals of a po P is an ideal of P, $\mathrm{Id}(P)$ is a dcpo. Recall that an element p of dcpo is *compact* if for every directed subset D, $p \leqslant \sup D$ implies that there exists $q \in S$ with $p \leqslant q$. Finally a partial order P is said to be *algebraic* (also algebraic domain) if it is a dcpo and for each $p \in P$ there exists a directed set D of compact elements with $\sup D = p$. It is easy to see that $\mathrm{Id}(P)$ is an algebraic partial order whose compact elements are exactly the principal ideals. Therefore every partial order is isomorphic to the partial order of compact elements of some algebraic partial order. In fact, conversely a partial order is algebraic if and only if it is isomorphic to the ideal completion of some partial order (see for example [Ern93]). We also refer the reader to the paper by Gehrke and Priestley [GP08, Proposition 2.1] for an abstract characterisation of the ideal completion of a partial order.

We now endow the ideal completion of a partial order with a topology. We do this by exploiting the fact that any dcpo can be equipped with several *intrinsic* topologies, i.e. topologies which can be defined by in terms of the partial order. We recall the relevant definitions and refer the reader to the textbook by Gierz et al. [Gie+03] for a more detailed presentation of these concepts.

For a dcpo P, a subset U of P is called *Scott open* if it is an upset and for every directed S, $\sup S \in U$ implies $S \cap U \neq \emptyset$. Scott open subsets of a dcpo P form a topology on P called the *Scott topology*.

The *lower topology* on a partial order P is generated by the base consisting of the finitely bounded downsets, namely the subsets of the form $P \setminus \uparrow F$ for finite $F \subseteq P$.

The *Lawson topology* on a dcpo P is the join of the Scott and the lower topology, it admits as a subbase the family of Scott open sets and sets of the form $P \setminus \uparrow p$, for $p \in P$, and as a base the sets of the form $U \setminus \uparrow F$ for U Scott open and F a finite subset of P.

For every partial order P let us endow $\mathrm{Id}(P)$ with the Lawson topology. It turns out that this topology on $\mathrm{Id}(P)$ admits another useful description.

Recall from Subsection 2.1.4 that the clopen subsets of the generalised Cantor space 2^P are finite unions of sets of the form

$$N(F, G) = \{X \subseteq P \mid F \subseteq X \text{ and } G \cap X = \emptyset\}$$

for finite subsets F and G of P. We write $\langle F \rangle$ instead of $N(F, \emptyset)$ and $\langle p \rangle$ instead of $\langle \{p\} \rangle$. In particular, the sets of the form $\langle p \rangle$ or $\langle p \rangle^{\complement}$ for $p \in P$ constitute a subbase for the topology of 2^P. As a matter of fact, the topology on 2^P is the Lawson topology associated with the partial order of inclusion on the subsets of P.

We now notice that the topology induced on $\mathrm{Id}(P)$ by 2^P is intrinsic:

Lemma 4.5. *Let P be a partial order. The Lawson topology on $\mathrm{Id}(P)$ coincides with the topology induced by 2^P.*

Proof. For $p \in P$, $\langle p \rangle \cap \mathrm{Id}(P)$ is an upset for inclusion. In addition, if S is a directed set of ideals and $p \in \bigcup S$ then for some $I \in S$ we have $p \in I$, i.e. $I \in \langle p \rangle$. Therefore $\langle p \rangle$ is Scott open. Moreover if $p \in P$, then $\langle p \rangle^{\complement} \cap \mathrm{Id}(P)$ is equal to $\{I \in \mathrm{Id}(P) \mid \downarrow p \nsubseteq I\}$, which is open in the lower topology by definition.

Conversely, let U be a Scott open set of $\mathrm{Id}(P)$. For every $I \in U$ since I is the directed union of the principal ideals $\downarrow p$ for $p \in I$, there must exist $p \in I$ with $\downarrow p \in U$, and therefore $I \in \langle p \rangle \cap \mathrm{Id}(P) \subseteq U$. Therefore U is open in the relative topology. Moreover for $I \in \mathrm{Id}(P)$, the lower open subbasic set

$\{J \in \mathrm{Id}(P) \mid I \nsubseteq J\}$ is equal to the set $\mathrm{Id}(P) \cap \bigcup_{p \in I} \langle p \rangle^{\complement}$ which is open in the relative topology. $\square$

In particular, since a subspace of the compact Hausdorff space 2^P is compact if and only if it is closed, we have the following.

Corollary 4.6. *The partial order* $\mathrm{Id}(P)$ *is compact in the Lawson topology if and only if* $\mathrm{Id}(P)$ *is closed as a subset of* 2^P*.*

Remark 4.7. Any partial order P can be equipped with the Alexandroff topology consisting in the upsets of P. The condition (ii) in Proposition 4.4 amounts to saying that the ideals are the irreducible non empty closed sets of P for the Alexandroff topology. Along this line, Hoffmann [Hof79] observed that the sobrification of a partial order P equipped with the Alexandroff topology coincide with the partial order $\mathrm{Id}(P)$ equipped with the Scott topology (see for example [Joh86, p. 291] for a proof).

4.1.2 The Cauchy ideal completion

We now regard a partial order P as a subset of the generalised Cantor space 2^P via the identification $p \mapsto {\downarrow} p$.

Definition 4.8. The topological closure of P inside 2^P equipped with the partial order of inclusion is denoted by $\mathrm{CId}(P)$ and referred to as the *Cauchy ideal completion* of the partial order P.

Erné and Palko [EP98] call a subset J of a partial order P a *Cauchy ideal* if for every finite $F \subseteq J$ and every finite $G \subseteq P \setminus J$ there exists $p \in P$ with $F \subseteq {\downarrow} p$ and $G \cap {\downarrow} p = \emptyset$. In our terminology, a subset $J \subseteq P$ is a Cauchy ideal if and only if for every finite $F, G \subseteq P$ with $J \in N(F, G)$ there exists $p \in P$ such that ${\downarrow} p \in N(F, G)$. Whence in other words a subset J of P is a Cauchy ideal exactly when $J \in \mathrm{CId}\, P$.

The term 'Cauchy ideal completion' is introduced by Erné and Palko [EP98] who define on any partial order P a uniformity U_P by taking as subbasic entourages the sets of the form

$$U_{\uparrow p} = \{(r, s) \in P \times P \mid p \leqslant r \leftrightarrow p \leqslant s\} = ({\uparrow} p \times {\uparrow} p) \cup (P \setminus {\uparrow} p \times P \setminus {\uparrow} p),$$

for $p \in P$. Now by the results of Section 3.2, the uniformity induced by the unique compatible uniformity on the compact Hausdorff space 2^P is described via the blocks of P.

Lemma 4.9. *Let P be a partial order viewed as a subset of 2^P via $p \mapsto \downarrow p$. The family of blocks of P is the Boolean algebra generated by the sets of the form $\uparrow p$ for $p \in P$.*

Proof. The family of clopen subsets of 2^P being the Boolean algebra generated by the sets of the form $\langle p \rangle$ for $p \in P$, it follows that the family of blocks of P is the Boolean algebra of subsets of P generated by the sets of the form $\langle p \rangle \cap P$ for each $p \in P$. Via the identification $p \mapsto \downarrow p$ we have

$$\langle p \rangle \cap P = \{q \in P \mid \downarrow q \in \langle p \rangle\} = \{q \in P \mid p \in \downarrow q\} = \uparrow p.$$

Therefore $\mathrm{Blocks}(P)$ is the Boolean algebra generated by the sets of the form $\uparrow p$ for $p \in P$. □

It follows that the uniformity associated with $\mathrm{Blocks}(P)$ (see Lemma 3.17) coincides with the uniformity U_P. In particular, it follows that the uniform completion of (P, U_P) is (uniformly isomorphic to) the topological closure of P in 2^P. Therefore as observed by Erné and Palko [EP98] the Cauchy ideals may also be viewed as the points of the uniform completion of (P, U_P). This justifies our use of their terminology. We also refer the reader to the paper by Erné [Ern01] where the idea of uniform completion and quasi-uniform completion of partial orders is further explored.

The following simple observation relates the ideal completion of a partial order with its Cauchy ideal completion.

Lemma 4.10. *Let P be a partial order.*

(i) *The set $\mathcal{D}(P)$ is closed in 2^P.*

(ii) *Every ideal is a Cauchy ideal.*

In particular, we have the following inclusions of partial orders:

$$P \subseteq \mathrm{Id}(P) \subseteq \mathrm{CId}(P) \subseteq \mathcal{D}(P).$$

Proof. (i) As we already observed in Subsection 2.1.4, the set $\mathcal{D}(P)$ is closed in 2^P since

$$\mathcal{D}(P) = \bigcap_{p \leqslant q} \langle q \rangle^{\complement} \cup \langle p \rangle.$$

(ii) Let I be an ideal of P and $N(F, G)$ – $F, G \in [Q]^{<\infty}$ – be any basic neighbourhood of I in 2^P. Since I is directed, $F \subseteq I$ implies that there exists $p \in I$ with $F \subseteq \downarrow p$. Since I is downward closed, $G \cap I = \emptyset$ implies

$G \cap {\downarrow} p = \emptyset$. Therefore ${\downarrow} p$ belongs to the neighbourhood $N(F, G)$. Since $N(F, G)$ was arbitrary, it follows that $I \in \mathrm{CId}(P)$.

Finally since $\mathcal{D}(P)$ is closed, we have $\mathrm{CId}(P) \subseteq \mathcal{D}(P)$ and by (ii) we have $\mathrm{Id}(P) \subseteq \mathrm{CId}(P)$. □

In general Cauchy ideals need *not* be directed (nor do they need to be non-empty), and $\mathrm{CId}(P)$ need not coincide with $\mathrm{Id}(P)$. Two particularly simple examples where $\mathrm{Id}(P)$ is not equal to $\mathrm{CId}(P)$ are given by the antichain $A = (\omega, =)$ and the descending chain $D = \omega^{\mathrm{op}}$ – the opposite of $(\omega, \leqslant)$. Indeed, while as we already noticed $\mathrm{Id}(A) = A$, the Cauchy ideal completion $\mathrm{CId}(A)$ is the antichain $A = \{\{n\} \mid n \in \omega\}$ with the addition of a minimal element $\emptyset$ which is also a limit point. Similarly we have $\mathrm{Id}(D) = D$, while $\mathrm{CId}(D)$ consists of D together with a minimal element which is also a limit point.

We now turn to the characterisation of the partial orders for which the ideal completion coincide with the Cauchy ideal completion. For any subset S of a partial order P let

$$S^{\uparrow} = \bigcap_{p \in S} {\uparrow} p = \{q \in P \mid \forall p \in S \quad p \leqslant q\},$$

the upset of *upper bounds* of S in P. Also, as we did for infinite subsets of the natural numbers, let us denote by $[P]^{<\infty}$ the set of finite subsets of a set P.

Definition 4.11. We say that a partial order P has *property* M if the set of upper bounds of any finite (possibly empty) subset of P admits a finite basis. In symbols, for all $F \in [P]^{<\infty}$ there exists $G \in [P]^{<\infty}$ such that $F^{\uparrow} = {\uparrow} G$.

Equivalently P has property M if P is finitely generated and for every $p, q \in P$ the upset ${\uparrow} p \cap {\uparrow} q$ is finitely generated.

Recall that by Proposition 2.6 (W4) a partial order P is WQO if and only if every upset is finitely generated. In particular every WQO has property M. However adding a minimal element $\perp$ to the antichain $(\omega, =)$ provides an example of a partial order with property M which is of course not WQO.

Notice however that while any subset of a WQO is WQO, the property M is not hereditary. Indeed removing the minimal element from $(\omega, =) \cup \{\perp\}$ results in the loss of the property M.

Remark 4.12. Bekkali, Pouzet, and Zhani [BPZ07] call a partial order P *up-closed* if ${\uparrow} p \cap {\uparrow} q$ is finitely generated for every $p, q \in P$. Their notion of up-closed differs from property M only in that $P = \emptyset^{\uparrow}$ need not be finitely generated.

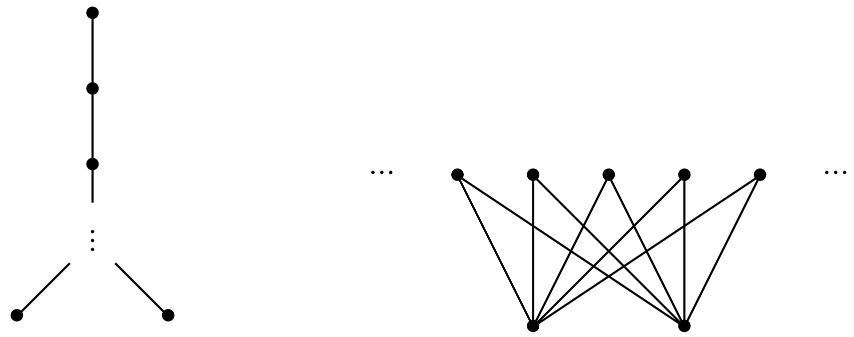


Figure 4.1: Examples of partial orders without property M.

Now observe that by Lemma 4.10 we have $\mathrm{Id}(P) = \mathrm{CId}(P)$ if and only if $\mathrm{Id}(P)$ is closed in 2^P. Moreover by Corollary 4.6 $\mathrm{Id}(P)$ is closed in 2^P if and only if $\mathrm{Id}(P)$ is compact in the Lawson topology. Finally it is folklore that a partial order P has property M if and only if $\mathrm{Id}(P)$ is compact in the Lawson topology (see e.g. [Gie+03, Corollary II-5.15, p.259 and Proposition I-4.10, p.118]). However to keep our exposition self-contained we provide the proof of the following proposition which can also be found in the paper by Bekkali, Pouzet, and Zhani [BPZ07].

Proposition 4.13. *For a partial order P, the following are equivalent.*

(i) *P has property M;*

(ii) $\mathrm{CId}(P) = \mathrm{Id}(P)$ *as subsets of* 2^P*;*

(iii) $\mathrm{Id}(P)$ *is compact in the Lawson topology.*

In particular, P has property M if and only if every Cauchy ideal of P is an ideal.

Proof. (i)→(ii): Suppose P has property M. For all $F \in [P]^{<\infty}$ there exists $G \in [P]^{<\infty}$ such that $F^{\uparrow} = \uparrow G$ and therefore

$$\bigcup_{r \in F^{\uparrow}} \langle r \rangle = \bigcup_{r \in G} \langle r \rangle$$

is clopen in 2^P as a finite union of clopen sets. Since $\emptyset^{\uparrow} = P$ is also finitely generated there exists a finite $B \subseteq P$ with $P = \uparrow B$. Now we can see that

$$\begin{aligned}\mathrm{Id}(P) =& \{I \in \mathcal{D}(P) \mid \forall F \in [P]^{<\infty} (F \subseteq I \to \exists r \in F^{\uparrow}\ r \in I) \wedge I \neq \emptyset\} \\ =& \mathcal{D}(P) \cap \bigcap_{F \in [P]^{<\infty}} \Big(\langle F \rangle^{\complement} \cup \bigcup_{r \in F^{\uparrow}} \langle r \rangle \Big) \cap \bigcup_{p \in B} \langle p \rangle.\end{aligned}$$

is closed in 2^P. Since $P \subseteq \mathrm{Id}(P) \subseteq \mathrm{CId}(P)$ by Lemma 4.10, we have $\mathrm{CId}(P) = \mathrm{Id}(P)$.

(ii)→(iii): By Lemma 4.10 again, we have $\mathrm{CId}(P) = \mathrm{Id}(P)$ if and only if $\mathrm{Id}(P)$ is closed in 2^P. Moreover by Corollary 4.6, $\mathrm{Id}(P)$ is closed in 2^P if and only if $\mathrm{Id}(P)$ is compact in the Lawson topology.

(iii)→(i): By Corollary 4.6, if $\mathrm{Id}(P)$ is compact in the Lawson topology, then $\mathrm{Id}(P)$ is closed in 2^P and therefore compact. In particular, the empty set – which is not an ideal by definition – does not belong to the closure of P and so $\emptyset$ admits a basic neighbourhood $N(\emptyset, G)$ for a finite subset G of P such that $\downarrow p \notin N(\emptyset, G)$ for all $p \in P$, i.e. $\uparrow G = P$, or in other symbols $\emptyset^{\uparrow} = \uparrow G$. Now let $F \subseteq P$ be non empty and finite. If $F^{\uparrow}$ is empty it is equal to $\uparrow \emptyset$, so suppose $F^{\uparrow}$ is non empty. Notice that the compact subspace $\langle F \rangle \cap \mathrm{Id}(P)$ is equal to the union of the basic open sets $\langle p \rangle$ for $p \in F^{\uparrow}$. Consequently there exists a finite $G \subseteq F^{\uparrow}$ with $\langle F \rangle \cap \mathrm{Id}(Q) = \bigcup_{q \in G} \langle q \rangle$ and it follows that $F^{\uparrow} = \uparrow G$, as desired. □

In particular when P is WQO, since P has property M, then the ideal completion $\mathrm{Id}(P)$ equipped with the Lawson topology coincides with the Cauchy ideal completion $\mathrm{CId}(P)$.

Notation 4.14. When P is WQO, we make a slight abuse of notation and denote by $\mathrm{Id}(P)$ the Cauchy ideal completion of P, or equivalently the ideal completion of P endowed with the Lawson topology. In this case, we refer to $\mathrm{Id}(P)$ as the *ideal space* of the WQO P.

For P WQO, the ideal space $\mathrm{Id}(P)$ is a partially ordered compact topological space. It actually enjoys further properties as we now see.

Recall that a point x of a topological space $\mathcal{X}$ is isolated in $\mathcal{X}$ if the singleton $\{x\}$ is open. A *limit point* of a topological space $\mathcal{X}$ is a point that is not *isolated*, i.e. for every neighbourhood U of x there is a point $y \in U$ with $y \neq x$. A topological space with no isolated points is called *perfect*. On the other extreme, a topological space is called *scattered* if it admits no perfect subspace.

Proposition 4.15 ([PS06])**.** *Let P be* WQO*. Then the space* $\mathrm{Id}(P)$ *is scattered. Moreover, an ideal I is isolated in* $\mathrm{Id}(P)$ *if and only if I is principal.*

Proof. We show that $\mathcal{D}(P)$ is scattered as a subspace of 2^P. Let $X \subseteq \mathcal{D}(P)$ be non empty. Since P is WQO, then $\mathcal{D}(P)$ is well-founded by Proposition 2.6 so there exists a $\subseteq$-minimal element D in X. Now since P has the finite basis property by Proposition 2.6 there exists a finite $F \subseteq P$ with $\uparrow F = P \setminus D$. It follows that $N(\emptyset, F) \cap X = \{D\}$, whence D is isolated in X. Since the property of being scattered is hereditary, it follows that $\mathrm{Id}(P)$ is scattered.

Now let $I \in \mathrm{Id}(P)$. If I is isolated, $\{I\}$ is open in $\mathrm{Id}(P)$, and since P is dense in $\mathrm{Id}(P)$, $I = \downarrow p$ holds for some $p \in P$ and so I is principal. Conversely for $p \in P$ let F be a finite basis for $P \setminus \downarrow p$. We have $N(\{q\}, F) \cap \mathrm{Id}(P) = \{\downarrow p\}$, so $\downarrow P$ is isolated in $\mathrm{Id}(P)$. $\square$

Moreover we have the following corollary of Lemma 2.12:

Proposition 4.16. *Let P be* WQO*. Every sequence $(I_n)_{n\in\omega}$ in* $\mathrm{Id}(P)$ *admits a subsequence $(I_j)_{j\in N}$ which converges to the ideal $\bigcup_{j\in N} I_j$ inside* $\mathrm{Id}(P)$*. In particular, every sequence $(p_n)_{n\in\omega} \subseteq P$ admits a subsequence $(p_n)_{n\in N}$ which converges to the ideal $\downarrow\{p_n \mid n \in N\}$ in* $\mathrm{Id}(P)$*.*

Proof. As P is WQO, $\mathrm{Id}(P)$ is a closed subset of $\mathcal{D}(Q)$ and therefore the first statement directly follows from Lemma 2.12. The second statement already follows from the more particular result stated in Fact 2.11. $\square$

To continue our study of the ideal space of a WQO it is instructive to explore the dual description of the Cauchy ideal completion $\mathrm{CId}(P)$ of an arbitrary partial order P. Importantly, this will lead us to recognise the ideal space of a WQO as the *profinite completion* or *Nachbin compactification* of this WQO.

4.1.3 Duality and Cauchy ideal completion

Zero-dimensional compact Hausdorff topological spaces play a prominent rôle in this thesis. Thanks to the seminal work of Stone [Sto36] these spaces are known to be intimately related to Boolean algebras. Their ordered analogues are the totally order-disconnected compact topological spaces, important examples of which are given in our context by the Cauchy ideal completion of a partial order. As discovered by Priestley [Pri72], totally-order disconnected compact spaces also possess a dual, algebraic, life as bounded distributive lattices. We now briefly recall the basic facts about these dualities and introduce some notations and conventions. We refer the reader to the monograph by Davey and Priestley [DP02] for details and proofs.

A topological space $\mathcal{X}$ equipped with a partial order $\leqslant$ is *totally order-disconnected* if given $x \nleqslant y$ in $\mathcal{X}$, there exists a clopen upset U such that $x \in U$ and $y \notin U$. A *Priestley space* is a totally order-disconnected compact topological space. For every set E the generalised Cantor space 2^E partially ordered by set-inclusion $\subseteq$ is a Priestley space. In fact up to isomorphism the Priestley spaces are exactly the closed subspaces of a space $(2^E, \subseteq)$ for some set E. In particular, the Cauchy ideal completion $\mathrm{CId}(P)$ of any partial order P is a Priestley space.

The morphisms between two Priestley spaces are the order-preserving and continuous maps or *Priestley maps*. The category of Priestley spaces with Priestley maps is denoted by **Pr**.

A *bounded distributive lattice*, henceforth simply a lattice, is a partial order L which admits a maximal element $\top$, a minimal element $\bot$, a greatest lower bound (infimum) $a \wedge b$ and a least upper bound (supremum) $a \vee b$ for every $a, b \in L$ and satisfies the distributivity law

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \quad \text{for every } a, b, c \in L.$$

Importantly, bounded distributive lattices are shown to form a variety of algebras in the signature $(\wedge, \vee, \bot, \top)$. A bounded lattice homomorphism, henceforth just a lattice homomorphism, is a map $h : L \to K$ preserving $\top$, $\bot$, $\wedge$ and $\vee$. We denote by **DLat** the category of (bounded distributive) lattices with (bounded) lattice homomorphisms.

A *prime filter* of a lattice L is a subset U of L such that $\bot \notin U$, $\top \in U$, $a, b \in U$ implies $a \wedge b \in U$, and for every $a, b \in L$, $a \vee b \in U$ implies $a \in U$ or $b \in U$. For a lattice L we denote by $\mathbf{D}(L)$ the set of prime filters of L partially ordered by *reverse inclusion* and endowed with the topology *Stone topology*, namely the topology generated by the sets of the form

$$\{U \in \mathbf{D}(L) \mid a \in U\} \text{ and } \{U \in \mathbf{D}(L) \mid a \notin U\}, \quad \text{for } a \in L.$$

Conversely for every Priesltey space $\mathcal{X}$ we denote by $\mathbf{E}(\mathcal{X})$ the lattice of clopen downsets of $\mathcal{X}$.

Moreover, to each lattice homomorphism $h : L \to K$ corresponds an order-preserving and continuous map $\mathbf{D}(h) : \mathbf{D}(K) \to \mathbf{D}(L)$ given by $\mathbf{D}(h)(U) = h^{-1}(U)$ for every prime filter U of the lattice K. Conversely to each order-preserving continuous map $f : \mathcal{X} \to \mathcal{Y}$ between Priestley spaces corresponds a lattice homomorphism $\mathbf{E}(f) : \mathbf{E}(\mathcal{Y}) \to \mathbf{E}(\mathcal{X})$ given by $\mathbf{E}(f)(D) = f^{-1}(D)$ for every clopen downset D of Y.

Theorem 4.17 (Priestley duality). *The functors $\mathbf{D} : \mathbf{DLat} \to \mathbf{Pr}$ and $\mathbf{E} : \mathbf{Pr} \to \mathbf{DLat}$ establish a duality of categories. In particular, $\mathbf{D}(\mathbf{E}(\mathcal{X}))$ and*

$\mathcal{X}$ are isomorphic as Priestley spaces and $\mathbf{E}(\mathbf{D}(L))$ and L are isomorphic as lattices.

Remark 4.18. We could have chosen to equip the set of prime filters of a lattice with the order of inclusion (instead of reverse inclusion) and have defined the dual of a Priestley space to be the lattice of clopen upsets (instead of clopen downsets) with the same result. While this convention makes some of the statements look nicer, we will regret in Theorem 4.20, for example, not to have chosen the other convention.

This duality encompasses two other famous dualities. Firstly, every finite partial order admits a unique topology, namely the discrete topology, which makes it a Priestley space. Priestley duality restricts to a duality between the category of finite partial orders with order-preserving maps and finite bounded distributive lattices: this is the content of Birkhoff duality [Bir37]. Secondly, Boolean algebras are particular examples of bounded distributive lattices. Since the prime filters of a Boolean algebra are the maximal filters, or ultrafilters, the partial order on the dual of a Boolean algebra under Priestley duality is discrete. The Priestley dual of a Boolean algebra, also called the *Stone dual*, is therefore simply a zero-dimensional compact Hausdorff space, or *Boolean space*. The restriction of Priestley duality to Boolean algebras on one side and Boolean spaces on the other is the celebrated Stone duality.

Importantly, we recall that Priestley duality turns sub-objects into quotients and vice versa:

Theorem 4.19. *For any lattice homomorphism $h : L \to K$ the following are equivalent.*

(i) *h is injective if and only if $\mathbf{D}(h)$ is surjective;*

(ii) *h is surjective if and only if $\mathbf{D}(h)$ is both an order embedding and a topological embedding.*

Towards the identification of the Priestley dual of the Cauchy ideal completion of a partial order P, we study a slightly more general situation where Priestley duality relates the taking of the topological closure with the algebraic generation of a lattice.

Let A and B be sets and $R \subseteq A \times B$ a binary relation from A to B. Let $R^{\leftarrow}(b) = \{a \in A \mid a\ R\ b\}$ for every $b \in T$. We denote by X_R the Priestley space consisting in the closure of the set $\{R^{\leftarrow}(b) \mid b \in B\}$ inside 2^A partially ordered by inclusion.

Let $R^{\rightarrow}(a) = \{b \in B \mid a\ R\ b\}$ for $a \in A$. We denote by L_R the bounded distributive lattice generated by $\{R^{\rightarrow}(a) \mid a \in A\}$ inside $(\mathcal{P}(B), \cap, \cup, \emptyset, B)$, namely the smallest family of subsets of B which contains $\{R^{\rightarrow}(a) \mid a \in A\}$, contains B and $\emptyset$, and is closed under both finite intersections and finite unions.

We denote by L_R^{op} the opposite of the lattice L_R. Observe that L_R^{op} is conveniently realised as the bounded distributive lattice generated by $\{\neg R^{\rightarrow}(a) \mid a \in A\}$ inside $(\mathcal{P}(B), \cap, \cup, \emptyset, B)$, where $\neg R^{\rightarrow}(a) = \{b \in B \mid \neg a\ R\ b\} = B \setminus R^{\rightarrow}(a)$.

Theorem 4.20 ([BPZ07]). *The lattice of clopen upsets of X_R is isomorphic to the lattice L_R, or dually X_R, is isomorphic to the space of prime filters of L_R ordered by inclusion and equipped with the Stone topology. Hence, with our convention, X_R is the Priestley dual of the lattice L_R^{op}.*

Bekkali, Pouzet, and Zhani [BPZ07] provide a direct proof of this fact. We now provide a 'fast proof' (cf. Figure 4.2) that we find to be conceptually more illuminating. It relies on the following fact. On the one hand, the space $(2^A, \subseteq)$ is dual to the free bounded distributive lattice on the set A of free generators. On the other hand, the 'free Priestley space' associated to the set B, namely the Stone–Czech compactification βB, is dual to the Boolean lattice $\mathcal{P}(B)$. This allows us to relate via Priestley duality the 'taking of the topological closure' and the process of 'generating a lattice'.

$$
\begin{tikzcd}
2^A & X_R \arrow[l, hook'] & \beta B \arrow[l, two heads] \arrow[ll, bend right, "\beta R^{\leftarrow}"'] \\
A \arrow[r, squiggly, no head] \arrow[d, hook] \arrow[drr, "R^{\rightarrow}"] & R \arrow[r, squiggly, no head] & B \arrow[u, hook] \arrow[ull, "R^{\leftarrow}"'] \\
F_{\mathbf{DLat}}A \arrow[r, two heads] \arrow[rr, bend right, "F_{\mathbf{DLat}}R^{\rightarrow}"'] & L_R \arrow[r, hook] & \mathcal{P}(B)
\end{tikzcd}
$$

Figure 4.2: Hamburger diagram: left vs right, algebra vs topology.

Proof. Consider the map $R^{\leftarrow} : B \longrightarrow 2^A$, $b \longmapsto R^{\leftarrow}(b)$ from the set B into the Priestley space 2^A partially ordered by inclusion. Now restricting the preimage map of $R^{\leftarrow}$ to the lattice F of clopen *upsets* of 2^A we obtain the lattice homomorphism $(R^{\leftarrow})^{-1} : F \to \mathcal{P}(B)$. The dual map of $(R^{\leftarrow})^{-1}$ is the unique continuous map $\beta R^{\leftarrow} : \beta B \to 2^A$ which extends $R^{\leftarrow}$ to the Stone–Czech compactification of the discrete space B. The image of $\beta R^{\leftarrow}$ inside 2^A is simply X_R.

The lattice of clopen *upsets* of $(2^A, \subseteq)$ is (isomorphic to) the free bounded distributive lattice $F_{\mathbf{DLat}}(A)$ on the set A of free generators and to every $a \in A$ corresponds the clopen *upset* $\{X \in 2^A \mid a \in X\}$. The lattice homomorphism $(R^{\leftarrow})^{-1}$ is therefore uniquely determined by its value on the sets $\{X \in 2^A \mid a \in X\}$, for $a \in A$, and we have

$$\begin{aligned}(R^{\leftarrow})^{-1}(\{X \in 2^A \mid a \in X\}) = \{b \in B \mid a \in R^{\leftarrow}(b)\}& \\ = \{b \in B \mid a\ R\ b\} = R^{\rightarrow}(a).&\end{aligned}$$

It follows that $(R^{\leftarrow})^{-1}$ is equal to the unique lattice homomorphism

$$F_{\mathbf{DLat}}(R^{\rightarrow}) : F_{\mathbf{DLat}}(A) \to \mathcal{P}(B)$$

which extends the map $R^{\rightarrow} : A \to \mathcal{P}(B)$. The image of $F_{\mathbf{DLat}}(R^{\rightarrow})$ in $\mathcal{P}(B)$ is of course the lattice L_R.

Summarising, the lattice homomorphism $F_{\mathbf{DLat}} R^{\rightarrow} : F_{\mathbf{DLat}}(A) \longrightarrow \mathcal{P}(B)$ is dual to the order-preserving continuous map $\beta R^{\leftarrow} : \beta B \longrightarrow 2^A$. Whence the lattice of clopen *upsets* of X_R is isomorphic to L_R. $\square$

We now apply Theorem 4.20 to the particular case of a reflexive, transitive and antisymmetric relation $\leqslant\ \subseteq P \times P$. Notice that the Priestley space $X_{\leqslant}$ is simply the Cauchy ideal completion $\mathrm{CId}(P)$. The lattice $L_{\leqslant}$ generated by the $\{\uparrow p \mid p \in P\}$ inside $(\mathcal{P}(P), P, \emptyset, \cap, \cup)$ is called the *tail lattice* [BPZ07] of P and is denoted by $\mathrm{TailLat}(P)$.

Theorem 4.21. *Let P be a partial order. Then the Cauchy ideal completion* $\mathrm{CId}(P)$ *is the Priestley dual of the opposite of* $\mathrm{TailLat}(P)$.

In fact, the property M really pertains to the dual lattice of $\mathrm{CId}(P)$.

Lemma 4.22. *The following are equivalent for a partial order P.*

(i) *P has property M;*

(ii) $\mathrm{TailLat}(P)$ *consists precisely in the finitely generated upsets of P;*

(iii) $\mathcal{D}_{fb}(P) = \mathrm{TailLat}(P)^{\mathrm{op}}$.

Proof. Firstly notice that for any po P the finitely generated upsets are closed under finite unions and contains the empty set – since $\emptyset = \uparrow\emptyset$ is finitely generated.

(i)$\leftrightarrow$ (ii): If P has property M and $F, G \subseteq Q$ are finite then it follows that the set $\uparrow F \cap \uparrow G = \bigcup_{p \in F, q \in G} \{p, q\}^{\uparrow}$ is finitely generated. Moreover $P = \emptyset^{\uparrow}$ is finitely generated if P has property M. The converse follows from the fact that by definition $F^{\uparrow} = \bigcap_{p \in F} \uparrow p$.

(ii)$\leftrightarrow$ (iii): Notice that the family $\mathcal{D}_{\mathrm{fb}}(Q)$ is obtained by complementation from the family of finitely generated upsets. $\square$

Whence for the partial orders which enjoy the property M we have the following.

Proposition 4.23. *Let P be a partial order with property M. Then the Cauchy ideal completion* $\mathrm{CId}(P)$*, which coincides with* $\mathrm{Id}(P)$ *equipped with the Lawson topology, is the Priestley dual of $\mathcal{D}_{fb}(P)$.*

We give the above mentioned isomorphism explicitly.

Proposition 4.24. *Let P be a partial order with property M. The map*

$$\begin{aligned} \mathbf{D}(\mathcal{D}_{fb}(P)) &\longrightarrow \mathrm{Id}(P) \\ U &\longmapsto I_U = \bigcap U = \{p \in P | P \setminus \uparrow p \notin U\} \end{aligned}$$

is an isomorphism between the Priestley dual of $\mathcal{D}_{fb}(P)$ and $\mathrm{Id}(P) = \mathrm{CId}(P)$ *whose inverse is given by*

$$\begin{aligned} \mathrm{Id}(P) &\longrightarrow \mathbf{D}(\mathcal{D}_{fb}(P)) \\ I &\longmapsto U_I = \{D \in \mathcal{D}_{fb}(P) \mid I \subseteq D\}. \end{aligned}$$

Proof. Let U be a prime filter of $\mathcal{D}_{\mathrm{fb}}(P)$. Let us first show that $\bigcap U = \{p \in P \mid P \setminus \uparrow p \notin U\}$. Clearly if $P \setminus \uparrow p \in U$, then $p \notin \bigcap U$. Conversely, if $p \notin \bigcap U$, then there exists $D \in U$ with $p \notin D$. Therefore $D \subseteq P \setminus \uparrow p$, and since U is a filter $P \setminus \uparrow p$ belongs to U too.

Notice that if P is the empty partial order, then both the dual of $\mathcal{D}_{\mathrm{fb}}(P) = \{\emptyset\}$ and $\mathrm{Id}(P)$ are empty. So suppose that P is not empty and that U is a prime filter of $\mathcal{D}_{\mathrm{fb}}(P)$. Clearly I_U is a downset of P. We show that I_U is an ideal of P. Since P has property M we have $P = \uparrow F$ for some non-empty finite $F \subseteq P$. Since $\emptyset = \bigcap_{r \in F} P \setminus \uparrow r$ does not belong to the prime filter U there exists $r \in F$ with $P \setminus \uparrow r \notin U$, and so $r \in \bigcap U$ and I_U is non empty.

Moreover I_U is directed as follows from Proposition 4.4 and the fact that U is prime.

The map $U \mapsto I_U$ is clearly an order embedding. To see it is continuous, observe that for every finite $F, G \subseteq P$ and every $U \in \mathbf{D}(\mathcal{D}_{\mathrm{fb}}(P))$:

$$I_U \in N(F,G) \quad \longleftrightarrow \quad \forall p \in F \quad P \setminus \uparrow p \notin U \quad \text{and} \quad \forall q \in G \quad P \setminus \uparrow q \in U.$$

As a continuous and injective map from a compact space to a Hausdorff space, $U \mapsto I_U$ is therefore a homeomorphism onto its image.

To see it is onto $\mathrm{Id}(P)$, let I be an ideal of P. Let us show that U_I is a prime filter of $\mathcal{D}_{\mathrm{fb}}(P)$. Clearly U_I is upward closed and closed under finite intersections. Since I is non empty, $\emptyset \notin U_I$. Finally the fact that U_I is prime follows from the directedness of I via Proposition 4.4 again. □

Returning to the case of a WQO, we recall that a partial order P is WQO if and only if every upset of P is finitely generated. Whence in this case, the lattice $\mathcal{D}_{\mathrm{fb}}(P)$ of finitely bounded downsets is simply equal to the lattice $\mathcal{D}(P)$ of downsets of P. Therefore when P is WQO, the Cauchy ideal completion is the Priestley dual of the lattice $\mathcal{D}(P)$. As a matter of fact, for an arbitrary partial order P the Priestley dual of the lattice $\mathcal{D}(P)$ of downsets turns out to be the *profinite completion of* P, also known as the *Nachbin order-compactification of* P.

4.1.4 Profinite completion, Nachbin order-compactification

We now describe another very natural completion of a partial order called the profinite completion. This completion is very natural from a category theory perspective. Though the material of this subsection is certainly considered folklore by experts, we were unable to find any reference. We therefore provide proofs for these basic results.

Let us denote by $\mathbf{Po}$ the category of partial orders together with order-preserving maps. The profinite completion of a partial order P is in a sense 'the best approximation of P among finite partial orders'. As it is, such a best approximation does not exist among finite partial orders, the reason being arguably that the category $\mathbf{Po}_{\mathrm{fin}}$ of finite partial orders is only finitely complete, but not complete. To remedy this and make sense of the above intuitive idea, one consider the pro-completion Pro-$\mathbf{Po}_{\mathrm{fin}}$ of the category $\mathbf{Po}_{\mathrm{fin}}$, which is somehow the completion of $\mathbf{Po}_{\mathrm{fin}}$ with respect to the property of admitting all (small) limits.

The category Pro-$\mathbf{Po}_{\mathrm{fin}}$ admits as objects directed diagrams $D : I \to \mathbf{Po}_{\mathrm{fin}}$ and the morphisms between $D : I \to \mathbf{Po}_{\mathrm{fin}}$ and $E : J \to \mathbf{Po}_{\mathrm{fin}}$ are given by the following double limit formula

$$\lim_J \lim_I \{f : D(i) \to E(j) \mid f \text{ is order-preserving}\}.$$

Fortunately the category Pro- $\mathbf{Po}_{\text{fin}}$ is equivalent to the concrete category $\mathbf{Pr}$ of Priestley spaces [Joh86, (VI 3.3), p. 248; Pri94; Spe72]. In this equivalence a directed diagram $D : I \to \mathbf{Po}_{\text{fin}}$ corresponds to the actual limit $\lim_I D(i)$ taken in the category of ordered topological spaces, regarding each finite partial order $D(i)$ as a discrete space, and the morphisms between profinite partial orders are realised as the order-preserving and continuous maps, the Priestley maps.

The profinite completion of a partial order P is – now formulated correctly – the best approximation (from the left) of P among the profinite partial orders, or equivalently, among the Priestley spaces. In formal terms, the profinite completion of partial orders $\mathsf{P} : \mathbf{Po} \to \mathbf{Pr}$ is the left adjoint to the forgetful functor $\mathsf{U} : \mathbf{Pr} \to \mathbf{Po}$ which simply drops the topology. We now describe concretely this functor.

For a partial order P we denote by $\mathcal{E}_P$ the set of all quasi-orders E on P which contains $\leqslant$ and whose partial order quotient is finite, i.e. the reflexive and transitive relations $E \subseteq P \times P$ such that

(1) for all $p, q \in P$, $p \leqslant q$ implies $p \; E \; q$,

(2) the quotient P/E by the equivalence relation ($p \; E \; q$ and $q \; E \; p$) has finitely many classes.

Now when partially ordered by inclusion $\mathcal{E}_P$ is directed, since $E, F \in \mathcal{E}_P$ implies that $E \cap F \in \mathcal{E}$. Moreover each time $E \subseteq F$ for $E, F \in \mathcal{E}_P$ there is a unique order-preserving and surjective map $\pi_{E,F} : P/E \to P/F$, $[p]_E \to [p]_F$. We let

$$\begin{aligned} D_P : \mathcal{E}_P &\longrightarrow \mathbf{Po}_{\text{fin}} \\ E &\longmapsto P/E \\ E \subseteq F &\longmapsto \pi_{E,F} : P/E \to P/F \end{aligned}$$

be the directed diagram of finite quotients of P.

Proposition 4.25. *Let P be a partial order and $\hat{P}$ be the limit in $\mathbf{Pr}$ of the diagram D_P, regarding finite partial orders as finite Priestley spaces. Then $\hat{P}$ is the Priestley dual of the lattice $\mathcal{D}(P)$.*

Proof. By Birkhoff duality, consider the diagram $D'_P : \mathcal{E}_P \to \mathbf{DLat}_{\text{fin}}$ dual to D_P, namely $D'_P(E) = \mathcal{D}(P/E)$ for every $E \in \mathcal{E}_P$ and $E \subseteq F \mapsto p^{-1}_{E,F} : \mathcal{D}(P/F) \to \mathcal{D}(P/E)$. Let D be the colimit of D'_P in $\mathbf{DLat}$. This distributive lattice is just the lattice $\mathcal{D}(Q)$ of downsets of Q. Indeed D is really the union of all downsets of Q of the form $p^{-1}(A)$ for an order-preserving map $p : Q \to R$ with R finite and A a downset of F. But for every downset L of Q the function $p_L : Q \to 2$ with $p_L(q) = 0 \leftrightarrow q \in L$ into $2 = \{0 < 1\}$ is order-preserving and $\chi_L^{-1}(\{0\}) = L$. By Priestley duality the colimit of D'_P in $\mathbf{DLat}$ is dual to the limit of D_P. Therefore $\hat{P}$ is the Priestley dual of $\mathcal{D}(P)$. □

From this description of $\hat{P}$ we get the following embedding

$$\begin{aligned} e : P &\longrightarrow \hat{P} \\ p &\longmapsto \hat{p} = \{D \in \mathcal{D}(P) \mid p \in D\}. \end{aligned}$$

We generally identify P with $e(P) = \{\hat{p} \mid p \in P\}$.

Lemma 4.26. *Let P be a partial order, then $\hat{P}$ is the closure of $\{\hat{p} \mid p \in P\}$ inside $2^{\mathcal{D}(P)}$ and so P is dense in $\hat{P}$.*

Proof (folklore). Suppose that $U \in 2^{\mathcal{D}(P)}$ belongs to the closure of $\{\hat{p} \mid p \in P\}$. Since $\{X \in 2^{\mathcal{D}(P)} \mid \emptyset \in X\}$ is open and $\emptyset \notin \hat{p}$ for all $p \in P$, $\emptyset \notin U$. If $D, E \in U$ and $D \cap E \notin U$, then U belongs to the basic open set $N(\{D, E\}, \{D \cap E\})$ of $2^{\mathcal{D}(P)}$, and so there exists $p \in P$ with $\hat{q} \in N(\{D, E\}, \{D \cap E\})$, i.e. $p \in D$, $p \in E$, and $p \notin D \cap E$, contradiction. Hence $D, E \in U$ implies $D \cap E \in U$. Next if $D \in U$ and $E \notin U$, then $U \in N(\{D\}, \{E\})$ and thus there exists $p \in P$ with $p \in D$ and $p \notin E$, so $D \nsubseteq E$. Finally if $D \cup E \in U$ and $D, E \notin U$, then $U \in N(\{D \cup E\}, \{D, E\})$ so there is $p \in P$ with $p \in D \cup E$ and $p \notin D$, $p \notin E$, a contradiction. So every $U \in 2^{\mathcal{D}(P)}$ which belongs to the closure of $e(P)$ is a prime filter on $\mathcal{D}(P)$.

Conversely let U be a prime filter on $\mathcal{D}(P)$ and $D_0, \dots, D_n, E_0, \dots E_m \in \mathcal{D}(P)$ be such that $U \in N(\{D_0, \dots, D_n\}, \{E_0, \dots, E_m\})$. Then since U is a prime filter, we have $D = \bigcap_{i=0}^n D_i \in U$ and $E = \bigcup_{j=0}^m E_j \notin U$, so necessarily $D \nsubseteq E$. Hence there exists $p \in D \setminus E$ and we have

$$\hat{p} \in N(\{D_0, \dots, D_n\}, \{E_0, \dots, E_m\}).$$

Therefore U belongs to the closure of $e(P)$. □

Proposition 4.27. *Let P be a partial order. For every Priestley space X and every order-preserving map $f : P \to X$ there exists a unique Priestley map $\hat{f} : \hat{P} \to X$ such that $g \circ e = \hat{f}$.*

$$
\begin{tikzcd}
 & \mathsf{U}(X) & & X \\
P \arrow[ur, "f"] \arrow[dr, hook, "e"'] & & & \\
 & \mathsf{U}(\hat{P}) \arrow[uu, "\mathsf{U}(\hat{f})"'] & & \hat{P} \arrow[uu, "\hat{f}"']
\end{tikzcd}
$$

Proof. Let $\mathsf{E}(X)$ be the lattice of clopen downsets of X. Since $f : P \to X$ is order-preserving the preimage map goes $f^{-1} : \mathcal{D}(X) \to \mathcal{D}(P)$ and restricts to a homomorphism $f^{-1} : \mathsf{E}(X) \to \mathcal{D}(P)$ whose Priesltey dual is an order-preserving continuous map $\hat{f} : \hat{P} \to X$. Clearly we have $\hat{f} \circ e = f$. Finally if $h : \hat{P} \to X$ is a Priestley map such that $h \circ e = f$, then $h{\restriction}_P = f = \hat{f}{\restriction}_P$. Since both h and $\hat{f}$ are continuous and agree the subset P of $\hat{P}$ which is dense by Lemma 4.26, we must have $\hat{f} = h$. □

This allows us to define the functor

$$
\begin{aligned}
\mathsf{P} : \mathbf{Po} &\longrightarrow \mathbf{Pr} \\
P &\longmapsto \hat{P} \\
(f : P \to Q) &\longmapsto (\hat{f} : \hat{P} \to \hat{Q})
\end{aligned}
$$

where $\hat{f}$ is the dual map of $f^{-1} : \mathcal{D}(Q) \to \mathcal{D}(P)$ under Priestley duality. By the previous proposition, this functor is left adjoint to the forgetful functor $\mathsf{U} : \mathbf{Pr} \to \mathbf{Po}$ as desired.

Notice that for a discrete partial order $(X, =)$ the profinite completion $\hat{X}$ is just the Stone-Čech compactification of the set X. Indeed $\mathcal{D}(X) = \mathcal{P}(X)$ and the Priestley dual of the complete Boolean algebra $\mathcal{P}(X)$ is simply its Stone dual. In particular, the profinite completion $\hat{A}$ of the antichain $A = (\omega, =)$ is the Stone-Čech compactification $\beta\omega$ of the natural numbers.

Actually, the profinite completion of a partial order enjoys a stronger universal property as notably proved by Bezhanishvili et al. [Bez+06].

Proposition 4.28 (Bezhanishvili et al. [Bez+06, Proposition 3.4])**.** *Let P be a partial order. For every compact Hausdorff space Y equipped with a partial order $\leqslant$ closed in $Y \times Y$ and every order preserving map $f : P \to Y$ there exists a unique continuous and order-preserving map $\hat{f} : \hat{P} \to Y$ such that $\hat{f} \circ e = f$.*

$$\begin{tikzcd}
 & Y \\
P \arrow[ur, "f"] \arrow[dr, hook, "e"'] & \\
 & \hat{P} \arrow[uu, "\hat{f}"']
\end{tikzcd}$$

This proposition shows that the profinite completion of a partial order P is also the Nachbin order-compactification [Nac65, p.103–104] of P equipped with the discrete topology.

We also notice that the profinite completion functor behaves nicely with respect to embeddings.

Lemma 4.29. *Let $f : P \to Q$ be an order-preserving map between posets. Then f is an order embedding if and only if $\hat{f} : \hat{P} \to \hat{Q}$ is both an order embedding and a topological embedding.*

Proof. By Theorem 4.19 it is enough to prove that $f : P \to Q$ is an order embedding if and only if $f^{-1} : \mathcal{D}(Q) \to \mathcal{D}(P)$ is surjective. Notice that if f is an embedding, then for every downset $D \in \mathcal{D}(P)$ we have $D = f^{-1}{\downarrow_Q} f(D)$ since

$$\begin{aligned}
p' \in f^{-1}{\downarrow_Q} f(D) \quad &\longleftrightarrow \quad \exists p \in D \quad f(p') \leqslant_Q f(p) \\
&\longleftrightarrow \quad \exists p \in D \quad p' \leqslant_P p \\
&\longleftrightarrow \quad p' \in D.
\end{aligned}$$

Conversely suppose $f^{-1} : \mathcal{D}(Q) \to \mathcal{D}(P)$ is surjective and let $p, p' \in P$ with $f(p) \leqslant f(p')$. There exists $D \in \mathcal{D}(Q)$ with $f^{-1}(D) = {\downarrow} p'$ and since $f(p) \leqslant f(p') \in D$ we have $p \in {\downarrow} p'$, i.e. $p \leqslant p'$. $\square$

Back to WQOs, we have obtained:

Corollary 4.30. *Let P be a partial order. If P is* WQO*, then the following are isomorphic as partially ordered topological spaces:*

(i) *the ideal completion* $\mathrm{Id}(P)$ *with the Lawson topology,*

(ii) *the Cauchy ideal completion* $\mathrm{CId}\, P$*,*

(iii) *the Priestley dual of* $\mathcal{D}(P)$*,*

(iv) *the profinite completion* $\hat{P}$*,*

(v) *the Nachbin order-compactification of P.*

If one wants to view the profinite completion $\widehat{P}$ of a WQO P as $\mathrm{Id}(P)$, then the action of the functor $\mathsf{P} : \mathbf{Po} \to \mathbf{Pr}$ on order-preserving maps between WQOs can be simply described as follows:

Lemma 4.31. *Let $f : P \to Q$ be an order preserving map between* WQO*s. The Priestley map $\mathsf{P}(f) : \widehat{P} \to \widehat{Q}$ viewed as a map $\mathsf{P}(f) : \mathrm{Id}(P) \to \mathrm{Id}(Q)$ is given by*

$$\mathsf{P}(f)(I) = \downarrow_Q\{f(p) \mid p \in I\} \quad \textit{for every } I \in \mathrm{Id}(P).$$

Proof. The Priestley map $\mathsf{P}(f) : \mathrm{Id}(P) \to \mathrm{Id}(Q)$ is the dual of the lattice homomorphism $f^{-1} : \mathcal{D}(Q) \to \mathcal{D}(P)$. Now an ideal $I \in \mathrm{Id}(P)$ corresponds to the prime filter $U_I = \{D \in \mathcal{D}(P) \mid I \subseteq D\}$ via the isomorphism given in Proposition 4.24. Then $V_I = \{E \in \mathcal{D}(Q) \mid f^{-1}(E) \in U_I\}$ is the prime filter of $\mathcal{D}(Q)$ which is the image of U_I by $\mathsf{P}(F) : \widehat{P} \to \widehat{Q}$. Now, by Proposition 4.24 again, the prime filter V_I corresponds to the ideal of Q given by $\{q \in Q \mid Q \setminus \uparrow q \notin V_I\}$. We therefore have

$$\begin{aligned}
\mathsf{P}(f)(I) =& \{q \in Q \mid Q \setminus \uparrow q \notin V_I\} \\
=& \{q \in Q \mid I \nsubseteq f^{-1}(Q \setminus \uparrow q)\} \\
=& \{q \in Q \mid \exists p \in I \ f(p) \notin Q \setminus \uparrow q\} \\
=& \{q \in Q \mid \exists p \in I \ f(p) \geqslant q\} \\
=& \downarrow_Q\{f(p) \mid p \in I\},
\end{aligned}$$

as desired. □

Remark 4.32. The *canonical extension* of a bounded distributive lattice is concretely defined as the lattice of downsets of its Priestley dual. However, it also admits an algebraic characterisation as the unique complete distributive lattice in which the lattice embeds in a dense and compact way (see for example the nicely written survey by Gehrke and Vosmaer [GV11]). For a WQO Q, the WQO character of $\mathrm{Id}(Q)$ is naturally related to the canonical extension of $\mathcal{D}(Q)$. Namely if Q is WQO, then $\mathrm{Id}(Q)$ is WQO if and only if the canonical extension of $\mathcal{D}(Q)$ is well-founded. Indeed $\mathrm{Id}(Q)$ is the Priestley dual of $\mathcal{D}(Q)$, so the lattice of downsets of $\mathrm{Id}(Q)$ is canonical extension of $\mathcal{D}(Q)$. Moreover $\mathrm{Id}(Q)$ is WQO if and only if $\mathcal{D}(\mathrm{Id}(Q))$ is well-founded by Proposition 2.6.

4.2 From well to better: A proof of Pouzet's conjecture

Henceforth Q stands for a WQO and we assume Q is a partial order. The *space of ideals* of Q is the partially ordered compact space described by one of the equivalent alternatives gathered in Corollary 4.30. The partial order Q is viewed as a subset of $\mathrm{Id}(Q)$ via the embedding $q \mapsto {\downarrow} q$. The space $\mathrm{Id}(Q)$, being a compact Hausdorff topological space, admits a unique uniformity which agrees with its topology. As a subset of $\mathrm{Id}(Q)$ the WQO Q is equipped with the induced uniformity. We henceforth refer to this uniformity on Q when we talk about a Cauchy, or uniformly continuous, super-sequence $f : F \to Q$ into a WQO.

Remark 4.33. It follows from Lemma 4.9 and Proposition 3.13 that a super-sequence $f : F \to Q$ into a WQO Q is Cauchy if and only if for every $q \in Q$ we have $f^{-1}({\uparrow} q) \in \mathrm{Blocks}(F)$.

Theorem 4.34 (with R. Carroy)**.** *Let Q be* WQO*. Then every super-sequence $f : F \to Q$ admits a Cauchy sub-super-sequence $f' : F' \to Q$ which therefore extends to a continuous map $\overline{f'} : \overline{F'} \to \mathrm{Id}(Q)$.*

Proof. As Q is WQO, the ideal space $\mathrm{Id}(Q)$ is a compact subspace of 2^Q. Now suppose first that Q is countable, then 2^Q is metrisable and therefore so is $\mathrm{Id}(Q)$. In this case, the statement follows from Theorem 3.25. In the general case, even though Q may not be countable, the set $P = \mathrm{Im}\, f$ always is. By Lemma 4.29, the embedding of P into Q extends to an embedding of $\mathrm{Id}(P)$ into $\mathrm{Id}(Q)$ both in the order and the topological sense. Therefore $f : F \to Q$ is uniformly continuous if and only if $f : F \to P$ is uniformly continuous. Since P is WQO as a subset of a WQO, the first case applies. □

Remark 4.35. In the previous proof, the only place where the hypothesis that Q is WQO, and not only has property M, is used is in the general case. This is when we conclude that P is WQO (or in fact that it has property M for that matter) from the fact that Q is WQO. This is indeed necessary since as we already noticed the property M is not hereditary.

As a corollary we have new characterisation of BQO:

Corollary 4.36. *A quasi-order Q is* BQO *if and only if Q is* WQO *and every Cauchy super-sequence is good.*

The *cofinality* of an ideal $I \in \mathrm{Id}(Q)$, denoted $\mathrm{cof}(I)$, is the least cardinal λ such that there exists a subset B of I of cardinality λ with $I = {\downarrow} B$. The principal ideals are the ideals of cofinality 1, they are the only ideals with finite cofinality. We denote respectively by $\mathrm{Id}_{\leqslant\omega}(Q)$ and $\mathrm{Id}_{\omega}(Q)$ the po of ideals of Q which have countable cofinality and the po of ideals with cofinality ω. Observe that $\mathrm{Id}_{\leqslant\omega}(Q) = Q \cup \mathrm{Id}_{\omega}(Q)$ and that $\mathrm{Id}_{\omega}(Q) \subseteq \mathrm{Id}^*(Q)$. The following should not come as a surprise.

Lemma 4.37. *If $f : F \to Q$ is a Cauchy super-sequence into a* WQO*, then the the image of the unique continuous extension $\bar{f} : \overline{F} \to \mathrm{Id}(Q)$ is contained in $\mathrm{Id}_{\leqslant\omega}(Q)$.*

Proof. Since $\overline{F}$ is metrisable, for every $s \in \overline{F}$ there exists a sequence $(s_n)_{n\in\omega} \subseteq F$ which converges to s. Then by continuity of $\overline{f}$ we have $\overline{f}(s) = \lim f(s_n)$. Then by Proposition 4.16, there is $N \in [\omega]^\infty$ such that $\overline{f}(s) = \bigcup_{n\in N} \downarrow f(s_n)$. Therefore $\bar{f}(s)$ has countable cofinality. □

Before proving the main result of this chapter we stop on our crucial example once again.

Example 4.38 (Rado's poset). Rado's partial order $\mathfrak{R}$ was defined in Example 2.15, it is given by the set $[\omega]^2$ partially ordered by :

$$\{m, n\} \leqslant \{m', n'\} \quad \longleftrightarrow \quad \begin{cases} m = m' \text{ and } n \leqslant n', \text{ or} \\ n < m'. \end{cases}$$

We claim that $\mathrm{Id}(\mathcal{R}) = \mathcal{R} \cup \{I_n \mid n \in \omega\} \cup \{\top\}$ where for all $n \in \omega$ $I_n = \bigcup_{n<k} \downarrow(n, k)$ and $\top = \mathcal{R}$. We have $(m, n) \leqslant I_k$ if and only if $m = k$ or $n < k$, and $a \leqslant \top$ for all $a \in \mathrm{Id}(\mathcal{R})$. The non principal ideals are the I_ns and $\top$. We show there are no other ideals. Let I be an ideal of $\mathcal{R}$. First suppose for all $k \in \omega$ there is an $(m, n) \in I$ with $k < m$, then $I = \top$. Suppose now that there exists $m = \max\{k \mid \exists l(k, l) \in I\}$. If there is infinitely many n such that $(m, n) \in I$ then $I = I_m$. Otherwise $I = \downarrow(m, n)$ for $n = \max\{l \mid (m, l) \in I\}$.

In Example 2.15 we showed that $\mathfrak{R}$ is WQO but not BQO as notably witnessed by the bad super-sequence $\mathrm{id} : [\omega]^2 \to \mathcal{R}$, which is the identity on the underlying sets. Using Remark 4.33, it is clear that id is Cauchy and therefore extends to a continuous map $\overline{\mathrm{id}} : [\omega]^{\leqslant 2} \to \mathrm{Id}(\mathcal{R})$. This continuous extension is simply given by $\overline{\mathrm{id}}(\{m\}) = I_m$ for every $m \in \omega$ and $\overline{\mathrm{id}}(\emptyset) = \top$ (see Figure 4.3). Now the restriction of $\overline{\mathrm{id}}$ to the barrier $[\omega]^1$ is a bad sequence in $\mathrm{Id}^*(\mathcal{R})$ witnessing the fact that it is not WQO. Hence this Cauchy bad super-sequence into $\mathcal{R}$ yields a bad super-sequence into the non principal ideals of $\mathcal{R}$.

Theorem 4.39. *Let $f : F \to Q$ be a Cauchy super-sequence into a* WQO *and $\bar{f} : \overline{F} \to \mathrm{Id}(Q)$ its continuous extension. If f is bad, then there exists a front*

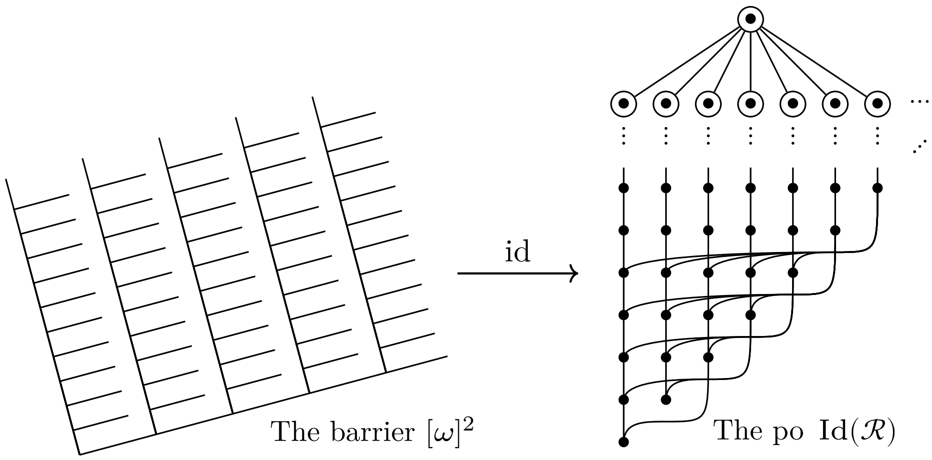


Figure 4.3: A bad Cauchy super-sequence into Rado's poset.

$G \subseteq \overline{F}$ such that the restriction $\bar{f}\restriction_G : G \to \mathrm{Id}^*(Q)$ is bad. Moreover $\bar{f}\restriction_G$ has image in $\mathrm{Id}_\omega(Q)$.

Proof. By going to $\check{f} : \check{F} \to Q$ if necessary by Proposition 2.53, we can assume that $f : F \to Q$ is spare. By Theorem 3.31 we can assume that $\overline{f} : \overline{F} \to \mathrm{Id}(Q)$ is normal. Since $\mathrm{Isol}(\mathrm{Id}(Q)) = Q$ by Proposition 4.15 we have $\Lambda_f = f^{-1}(\mathrm{Id}^*(Q))$. Moreover by Lemma 3.36 we obtain $f^{-1}(\mathrm{Id}^*(Q)) = \overline{F} \setminus F$. If $f^{-1}(\mathrm{Id}^*(Q))$ is empty, then F is trivial and thus f is good, a contradiction. Otherwise $f^{-1}(\mathrm{Id}^*(Q))$ is the closure of a front G on $X = \bigcup F$ and by Lemma 3.37 we have

$$F = \{s \cup \{n\} \mid s \in G \text{ and } n \in X/s\}.$$

If G is trivial then $F = [X]^1$, since Q is WQO, then f is good, a contradiction again.

Otherwise G is not trivial, and consider $g = \bar{f}\restriction_G : G \to \mathrm{Id}^*(Q)$. By contradiction, suppose there is no front $G' \subseteq G$ such that the restriction $g\restriction_{G'} : G' \to \mathrm{Id}^*(Q)$ is bad. Hence by Corollary 2.63 there exists a front $G' \subseteq G$ such that the restriction $g\restriction_{G'} : G' \to Q$ is perfect. Let $s \in G'$ be minimal for $\subseteq$ in G' and $Y = \bigcup G'$. By continuity we have $g(s) = \lim_{n\in Y/s} f(s \cup \{n\})$ and by Proposition 4.16 there is $Z \in [Y/s]^\infty$ such that $g(s) = \{q \in Q \mid \exists n \in Z\ q \leqslant f(s \cup \{n\})\}$. For $n_0 = \min Z$ there exists $t \in G'$ with $t \sqsubset {}_*s \cup \{n_0\} \cup Y/n_0$. By the minimality of s in G' for the inclusion, we have ${}_*s \cup \{n_0\} \sqsubseteq t$. Again $g(t) = \lim_{n\in Y/t} f(t \cup \{n\})$ and by Proposition 4.16 there is $Z' \in [Y/t]^\infty$ with $g(t) =\{q \in Q \mid \exists m \in Z'\ q \leqslant f(s \cup \{m\})\}$. Since $s \lhd t$ in G' and

g is perfect, we have $g(s) \subseteq g(t)$. Therefore there exists $m_0 \in Z'$ with $f(s \cup \{n_0\}) \leqslant f(t \cup \{m_0\})$. Since $s \cup \{n_0\} \lhd t \cup \{m_0\}$, f is good. This contradiction terminates the proof. □

As a direct corollary we have the proof of the conjecture by Pouzet [Pou78].

Theorem 4.40 (with R. Carroy)**.** *Let Q be* WQO*. If* $\mathrm{Id}_\omega(Q)$ *is* BQO*, then Q is* BQO*. In particular if* $\mathrm{Id}^*(Q)$ *is* BQO*, then Q is* BQO*.*

The following classes of WQOs are sometimes considered as approximations of the concept of BQO.

Definition 4.41. Let Q be a quasi-order and $1 \leqslant \alpha < \omega_1$. We say that Q is α-BQO if and only if every super-sequence $f : F \to Q$ with $\operatorname{rk} F \leqslant \alpha$ is good.

Clearly a qo is WQO if and only if it is 1-BQO, and it is BQO if and only if its α-BQO for every $\alpha < \omega_1$.

Remark 4.42. Marcone [Mar94], Pouzet and Sauer [PS06], for example, use a different definition of α-BQO which is easily seen to be equivalent to ours.

A finer utilisation of Theorem 4.39 yields the following:

Theorem 4.43. *Let Q be a quasi-order.*

(i) *For every $\alpha < \omega_1$, if Q is $(\alpha+1)$-*BQO*, then $\mathcal{D}(Q)$ is α-*BQO *and so* $\mathrm{Id}^*(Q)$ *is α-*BQO *too.*

(ii) *If Q is* WQO*, then for every $\alpha < \omega_1$,* $\mathrm{Id}_\omega(Q)$ *is α-*BQO *implies that Q is $(1+\alpha)$-*BQO*.*

(iii) *If Q is* WQO *then for every $n < \omega$, Q is $(1+n)$-*BQO *if and only if* $\mathrm{Id}_\omega(Q)$ *is n-*BQO*.*

Proof. (i) By contraposition, suppose $f : F \to \mathcal{D}(Q)$ is a bad super-sequence from some front F. A bad super-sequence $f^2 : F^2 \to Q$ from the front

$$F^2 = \{s \cup t \mid s, t \in F \text{ and } s \lhd t\}$$

is obtained by choosing for every $s, t \in F$ with $s \lhd t$ some $f(s \cup t) \in f(s) \setminus f(t)$. By observing that for each ray F^2_n, the tree $T(F^2_n)$ is included in the union $T(F) \cup T(F_n)$, we see that $\operatorname{rk} F^2 \leqslant \operatorname{rk} F + 1$.

(ii) By contraposition, suppose Q is WQO and there is bad super-sequence from a front F into Q, then by Theorem 4.39 there exists a bad super-sequence from some front G with $G \subseteq \overline{F} \setminus F$ into $\mathrm{Id}_\omega(Q)$. By Lemma 3.34 we have $1 + \operatorname{rk} G \leqslant \operatorname{rk} F$, so if $\operatorname{rk} F \leqslant 1 + \alpha$ then $\operatorname{rk} G \leqslant \alpha$.

(iii) Since for a finite ordinal n we have $n+1 = 1+n$, the statement follows from (i) and (ii). □

Remark 4.44. In an attempt to prove the converse of Theorem 4.43 (ii), one could try the following. For a bad super-sequence $f : F \to \mathrm{Id}_\omega(Q)$, one could consider a map $f^+ : F^+ \to Q$ defined on the front

$$F^+ = \{s \cup \{n\} \mid s \in F \text{ and } n \in \omega/s\}$$

by choosing a strictly increasing sequence $(q^s_n)_{n\in\omega/s}$ with $\downarrow\{q^s_n \mid n \in \omega/s\} = f(s)$ and letting $f(s \cup \{n\}) = q^s_n$ for every $s \in F$ and $n \in \omega/s$. Notice that $\operatorname{rk} F^+ = 1 + \operatorname{rk} F$ as desired. However assuming that f^+ is perfect is not sufficient to prove that f is good, as the following example shows.

Consider the super-sequence $f^+ : [\omega]^2 \to \mathfrak{R}$ into Rado's partial order given by $f^+(m,n) = (2m, m+n)$. The super-sequence f^+ is perfect since whenever $m < n < k$ we have $(2m, m+n) \leqslant (2n, n+k)$ in $\mathfrak{R}$. Using Remark 4.33, f^+ is easily seen to be Cauchy and therefore extends to a continuous $\bar{f} : [\omega]^{\leqslant 2} \to \mathrm{Id}(\mathfrak{R})$. But clearly the restriction $f = \bar{f}\restriction_{[\omega]^1} \to \mathrm{Id}^*(\mathfrak{R})$ is bad.

This observation is a motivation to seek to strengthen the notion of 'perfect super-sequence' in the conclusion of the implication 'if there is no bad sequence in Q, then every super-sequence admits a perfect sub-super-sequence' while maintaining the validity of this fact. The results presented in Section 2.3 can be seen as a first attempt in this direction.

Although some ingredients seem yet to be missing, we make the following conjecture. Notice that it differs from the claim made by Pouzet and Sauer [PS06, Theorem 2.17].

Conjecture 2. *Let Q be* WQO *and $\alpha < \omega_1$. Then Q is $(1+\alpha)$-*BQO *if and only if $\mathrm{Id}_\omega(Q)$ is α-*BQO.

4.3 Corollaries

4.3.1 Finitely many non principal ideals

The first corollary of Theorem 4.40 that we mention is:

Corollary 4.45. *If Q is* WQO *and $\mathrm{Id}^*(Q)$ is finite, then Q is* BQO.

This result is due to Pouzet [Pou78] and a direct proof is presented by Fraïssé [Fra00, Chapter 7, 7.7.8].

This first simple corollary already allows us to prove the following proposition, a particular case of which was used by Carroy [Car13].

Proposition 4.46. *Let $\varphi : \omega \to \omega$ be progressive, i.e. such that $n \leqslant \varphi(n)$ for every $n \in \omega$. Then the partial order $\leqslant_\varphi$ on ω defined by*

$$m \leqslant_\varphi n \quad \leftrightarrow \quad m = n \text{ or } \varphi(m) < n.$$

is a better-quasi-order.

Proof. Let $g : \omega \to \omega$ be any sequence. Then either g is bounded in the usual order and so g is good for $\leqslant_\varphi$, or g is unbounded in the usual order and so there exists n such that $g(n) > \varphi(g(0))$ and so g is good for $\leqslant_\varphi$. Hence $(\omega, \leqslant_\varphi)$ is WQO.

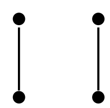
Now let I be a non principal ideal in $(\omega, \leqslant_\varphi)$. In particular I is an infinite subset of ω, so for every $m \in \omega$ there exists $n \in I$ such that $\varphi(m) < n$ and so $m \leqslant_\varphi n \in I$. Therefore $I = \omega$ and so there is exactly one non-principal ideal of $(\omega, \leqslant_\varphi)$. It follows by Corollary 4.45 that $(\omega, \leqslant_\varphi)$ is BQO. $\square$

4.3.2 Interval orders

We now turn to the case of a WQO Q such that $\mathrm{Id}^*(Q)$ is a well order. Since by Proposition 2.54 well orders are BQO, such quasi-orders are BQO by Theorem 4.40.

Observe that when Q is WQO, since ideals are downsets and $\mathcal{D}(Q)$ is well-founded, $\mathrm{Id}^*(Q)$ is well-founded too. Henceforth, if Q is WQO then $\mathrm{Id}^*(Q)$ is linearly ordered if and only if $\mathrm{Id}^*(Q)$ is a well order.

What are the quasi-orders whose non principal ideals are linearly ordered? Well, assume Q is a quasi-order and that $I, J \in \mathrm{Id}^*(Q)$ are incomparable for inclusion. Let $p \in I \setminus J$ and $q \in J \setminus I$. Then p is incomparable with q. Forbidding antichains of size 2 in Q is simply asking that Q is a linear order, and of course well orders are BQO. But we can do better: since I and J are non principal, there are $p' \in I$ with $p < p'$ and $q' \in J$ with $q < q'$. The restriction of the quasi-order on Q to $\{p, q, p', q'\}$ is isomorphic to the po

$2 \oplus 2 =$ 

and therefore $2 \oplus 2$ embeds into Q. We are naturally led to following definition which appears frequently in the literature.

Definition 4.47. A partial order P is an *interval order* if the partial order $2 \oplus 2$ does not embed into P. In other words for every $p, q, x, y \in P$, $p < x$ and $q < y$ imply $p < y$ or $q < x$.

The preceding discussion yields the following which is already stated by Pouzet and Sauer [PS06].

Theorem 4.48. *An interval order is* BQO *if and only if it is* WQO.

Notice that this theorem can be rephrased as follows: a partial order P such that neither $(\omega, =)$, nor ω^{op} (the opposite of ω), nor $2 \oplus 2$ embeds into P is a better-quasi-order.

According to Fishburn and Monjardet [FM92], the notion of *interval order* was first studied by the twenty-years-old Norbert Wiener [Wie14] who credits Bertrand Russell for suggesting the subject. Wiener was later acknowledged as the originator of cybernetics [CS06]. The reverse mathematics of interval orders is studied by Marcone [Mar07].

For $p \in P$, let $\mathrm{Pred}(p) = \{q \in P \mid q < p\}$. It is easy to see that a partial order P is an interval order if and only if the set $\{\mathrm{Pred}(p) \mid p \in P\}$ is linearly ordered by inclusion.

The terminology 'interval order' was introduced by Fishburn [Fis70] and stems from the following characterisation.

A non trivial *closed interval* of a partial order Q is a set of the form $[a, b] = \{q \in Q \mid a \leqslant q \leqslant b\}$ for some $a, b \in Q$ with $a < b$. We partially order the set $\mathrm{Int}(Q)$ of non trivial closed intervals of Q by $[a, b] \leqslant [c, d]$ if and only if $a = c$ and $b = d$ or $b \leqslant c$.

For a partial order P let us say that a map $I : P \to \mathrm{Int}(Q)$ is an *interval representation* of P in Q if for every $x, y \in P$ we have $x < y \leftrightarrow I(x) < I(y)$.

Let us first see that any partial order P admits an interval representation. Let $\mathrm{Pred}^+(p) = \bigcap_{p<x} \mathrm{Pred}(x)$ and

$$Q_P = \{\mathrm{Pred}(p) \mid p \in P\} \cup \{\mathrm{Pred}^+(p) \mid p \in P\}$$

be partially ordered by inclusion.

Proposition 4.49 ([Bog93])**.** *Let P be a partial order. The map*

$$\begin{aligned} I : P &\longrightarrow \mathrm{Int}(Q_P) \\ p &\longmapsto I_p = \Big(\mathrm{Pred}(p), \mathrm{Pred}^+(p)\Big) \end{aligned}$$

is an interval representation of P in Q_P.

Proof. First observe that for every $p \in P$ we have $\mathrm{Pred}(q) \subset \mathrm{Pred}^+(q)$ since $q < p$ imply $q \in \mathrm{Pred}(x)$ for all $x > p$, and in fact $p \in \mathrm{Pred}^+(p) \setminus \mathrm{Pred}(p)$. So I is well defined. If $p < q$, then $\mathrm{Pred}^+(p) = \bigcap_{p<x} \mathrm{Pred}(x) \subseteq \mathrm{Pred}(q)$, and so $I_p < I_q$. Conversely if $I_p < I_q$, then $\mathrm{Pred}^+(p) \subseteq \mathrm{Pred}(q)$ and since $p \in \mathrm{Pred}^+(p)$ we have $p < q$. Hence I is an interval representation of P. □

The following is a slight generalisation of a theorem by Fishburn [Fis70]. The proof we give is due to Bogart [Bog93].

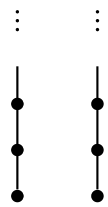
Proposition 4.50. *A partial order P is an interval order if and only if there exists an interval representation of P in some linear order.*

Proof. Suppose $I : P \to \mathrm{Int}(L)$ is an interval representation of P in a linear order L and let $p_0 < p_1$ and $q_0 < q_1$ in P. If $I(p_i) = [l_i, r_i]$ and $I(q_i) = [m_i, s_i]$ then $r_0 \leqslant l_1$ and $s_0 \leqslant m_1$. Since L is linearly ordered, either $r_0 \leqslant m_1$ and so $p_0 < q_1$, or $m_1 \leqslant r_0$ and so $q_0 < p_1$. Therefore P is an interval order.

Conversely, suppose P is an interval order. By Proposition 4.49, it suffices to prove the Q_P is linearly ordered. But $\{\mathrm{Pred}(p) \mid p \in P\}$ is linearly ordered and $\mathrm{Pred}^+(p) = \bigcap_{p<x} \mathrm{Pred}(x)$ is incomparable for the inclusion with some $X \in Q_P$ if and only if $\mathrm{Pred}(x)$ is incomparable with X for some $x > p$. □

4.3.3 Classes of better-quasi-orders via forbidden patterns

In fact continuing the above discussion we find that for any qo Q, $\mathrm{Id}^*(Q)$ is linearly ordered if and only if the po

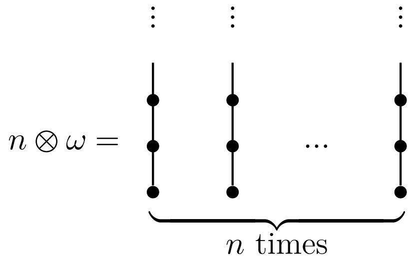
$\omega \oplus \omega =$ 

does not embed into Q. We therefore have the following:

Theorem 4.51. *If neither $(\omega, =)$, nor ω^{op}, nor $\omega \oplus \omega$ embed into Q, then Q is* BQO.

Suppose now for a partial order P that there exists a natural number n such that the size of every antichain of P is bounded by n. Then, by a theorem due to Dilworth [Dil50], for A an antichain of maximum size, say n, there exist subsets P_i, $i \in n$, such that $|P_i \cap A| = 1$, P_i is linearly ordered and $\bigcup_{i \in n} P_i = P$ (see also [Fra00, 4.14.1, p. 141]). In particular, if P is further assumed to be well-founded, then P is BQO as a finite union of well orders.

Continuing further the discussion of the previous subsection, we see that if there exists an antichain A of size n among the non principal ideals of a qo Q, then the partial order



embeds into Q. Indeed, assume that $\{I_i \mid i \in n\}$, $n \geqslant 2$ is an antichain of non principal ideals of a qo Q. For each $i \in n$ and every $j \in n$ with $i \neq j$, since $I_i \nsubseteq I_j$ we can pick $q_j \in I_i \setminus I_j$ and by the fact that I_i is directed there is $q^i \in I_i$ with $q^i \notin I_j$ for every $j \neq i$. Now since each I_i is non principal there exists a strictly increasing sequence $(q^i_k)_{k \in \omega}$ in I_i with $q^i_0 = q^i$. This clearly yields and embedding of $n \otimes \omega$ into $\bigcup_{i \in n} I_i$. Therefore

Theorem 4.52. *Let $n \geqslant 1$. If neither $(\omega, =)$, nor ω^{op}, nor $n \otimes \omega$ embed into Q, then Q is* BQO.

In this theorem, for each $n \geqslant 1$, we have a class of BQO which is defined by finitely many forbidden patterns. Examples of classes of BQOs defined by mean of forbidden patterns – left alone by finitely many – are quite rare. In fact to our knowledge the previous theorem is the best result of this sort.

5 A Wadge hierarchy for second countable spaces

Our last chapter is of a more applied nature. We are interested in quasi-ordering the subsets of a topological space according to their complexities. Among the properties of such a quasi-order the following are arguably wished for. It should agree with an a priori idea of topological complexity, in particular it should refine the classical hierarchies of topological complexity. Moreover it should be as fine as possible while still being WQO or even BQO – at least on the Borel subsets.

The Wadge quasi-order on the Baire space ω^ω satisfies all these properties and even more. But the fact that the Wadge quasi-order is well-founded on Borel subsets of ω^ω is not at all straightforward. It relies on the determinacy of certain infinite games and this result is actually best seen as an immediate corollary of a theorem on BQOs obtained by van Engelen, Miller, and Steel [vEMS87, Theorem 3.2]. Elaborating on Chapter 2, we start this chapter by presenting a slight generalisation of this theorem in a way which makes it appear as an extension of the very idea underlying the definition of BQO.

We then define a notion of reducibility based on relatively continuous relations. We show that this notion of reducibility satisfies the above mentioned properties and generalises the Wadge quasi-order to virtually all second countable T_0 spaces.

This chapter is based on an article [Peq15] published by the author in *Archive for Mathematical Logic*.

5.1 The van Engelen–Miller–Steel theorem

In Chapter 2, we started from the idea that a qo Q is BQO if the quasi-order $V^*(Q)$ of non empty sets over Q is WQO. We then found a convenient definition of BQO by observing that any bad sequence in $V^*(Q)$ yields a multi-sequence into Q of a particular kind, which we identified as the bad multi-sequences. Pushing the same idea a little further we showed that when Q is BQO, then so is $V^*(Q)$. We then observed in Subsection 2.2.5 that the restriction to the quasi-order $H^*_{\omega_1}(Q)$ of hereditarily countable non-empty subsets over Q already contained all the required information. In particular, we showed in Theorems 2.58 and 2.59 that Q is BQO if and only if $H^*_{\omega_1}(Q)$ is WQO if and only if $H^*_{\omega_1}(Q)$ is well-founded. We now describe $H^*_{\omega_1}(Q)$ in slightly different terms before we greatly generalise this construction in a way that is crucial to the present chapter.

We can endow a set A with the discrete topology in which every subset of A is open. This space A admits the compatible metric $\delta(a,b) = 1$ whenever $a \neq b$. The set A^ω of infinite sequences on A, viewed as the product space of infinitely many copies of A, is also metrisable: we use the compatible ultrametric $d(x,y) = 2^{-n}$ if $x \neq y$ and n is the least number with $x_n \neq y_n$. When A is countable, A^ω is a Polish space, i.e. a separable completely metrisable space. Important examples are given by the Cantor space 2^ω and the Baire space ω^ω.

Recall from Definitions 2.28 that by a tree on a non-empty set A we mean a set $T \subseteq A^{<\omega}$ of finite sequences on A which is closed under prefixes, i.e. $u \sqsubseteq v$ and $v \in T$ imply $u \in T$. We say that a tree T on A is *pruned* if for every $u \in T$ there exists $v \in T$ with $u \sqsubset v$. For a tree T on A, we let

$$[T] = \{x \in A^\omega \mid \text{for every } n \in \omega,\, x{\restriction}_n \in T\}$$

denote the set of infinite branches of T. It is well-known that the map $T \mapsto [T]$ is a one-to-one correspondence between pruned trees on A and closed subsets of A^ω.

For $X \in V^*(Q)$ recall that $\operatorname{tc}_{\mathrm{Q}} X$ is the smallest transitive set containing $\{X\}$ when elements of Q are treated as urelements or atoms, namely each element of Q contains no element and is different from the empty set. Notice that in particular for $q \in Q$ we have $\operatorname{tc}_{\mathrm{Q}}(q) = \{q\}$. Therefore by the axiom of countable choice we have $X \in H^*_{\omega_1}(Q)$ if and only if $X \in V^*(Q)$ and $\operatorname{tc}_{\mathrm{Q}}(X)$ is countable.

For every $X \in H^*_{\omega_1}(Q)$ we define a non-empty pruned tree T_X on $\operatorname{tc}_{\mathrm{Q}}(X)$ by letting

(1) if $X \in Q$, then $T_X = \{X\}^{<\omega}$,

(2) if $X \notin Q$, then T_X is the the set of finite sequences s on $\operatorname{tc}_{\mathrm{Q}}(X)$ such that

$$s = \emptyset \quad \text{or} \quad \Big[s_0 \in X \text{ and } \forall i < |s| - 1 \\ (s_i \ni s_{i+1} \text{ or } (s_i \in Q \text{ and } s_i = s_{i+1}))\Big].$$

Next we define a map $l_X : [T_X] \to Q$ by letting for every $\alpha \in [T_X]$

$$l_X(\alpha) = q \quad \longleftrightarrow \quad \exists n \in \omega \quad \alpha_n = q.$$

By the axiom of foundation l_X is defined on the whole of $[T_X]$. Moreover l_X is locally constant, so $l_X^{-1}(q)$ is open in $[T_X]$ for every $q \in Q$ and l_X has a countable image.

For every $X \in H^*_{\omega_1}(Q)$ the map $l_X : [l_X] \to Q$ is – up to a bijection between $\mathrm{tc}_{\mathrm{Q}}(X)$ and ω – a locally constant map from a closed subset of ω^ω into Q.

We now generalise $H^*_{\omega_1}(Q)$ in the following way. Recall that the family of Borel subsets of a topological space $\mathcal{X}$ is the smallest Boolean algebra of subsets of $\mathcal{X}$ containing the open sets and closed under countable union and complementation – and hence under countable intersection. Recall that if $\mathcal{Y}$ is a subspace of $\mathcal{X}$ then the family of Borel sets of $\mathcal{Y}$ consists of the intersection with $\mathcal{Y}$ of some Borel set of $\mathcal{X}$.

Definition 5.1. Let Q be a quasi-order. A map $l : D \to Q$ from a Borel subset D of ω^ω is called a *Borel Q-labelling function on D* if

(1) for every $q \in Q$, the set $l^{-1}(q)$ is Borel in D,

(2) the image $\operatorname{Im} l$ of l is countable.

We refer to D as the domain of l and denote it by $\operatorname{dom} l$. We let $\mathfrak{L}^B(Q)$ be the set of all Borel Q-labelling functions on some Borel subset of ω^ω.

For every $l_0 : D_0 \to Q$ and $l_1 : D_1 \to Q$ in $\mathfrak{L}^B(Q)$ we define a two player game with perfect information $G(l_0, l_1)$ as follows (see Figure 5.1). The players I and II choose natural numbers alternatively. Player I starts by choosing some $\alpha_0 \in \omega$, then Player II chooses some $\beta_0 \in \omega$, then it is again the turn of Player I to choose some $\alpha_1 \in \omega$, so on and so forth. Both players eventually produce $\alpha \in \omega^\omega$ and $\beta \in \omega^\omega$ respectively. Player II wins the play (α, β) if and only if the following condition holds:

$$\alpha \in D_0 \text{ implies } \big(\beta \in D_1 \text{ and } l_0(\alpha) \leqslant l_1(\beta)\big).$$

Player I wins the play (α, β) if and only if Player II does not win, namely when the following condition holds:

$$\alpha \in D_0 \text{ and } \big(\beta \in D_1 \text{ implies } l_0(\alpha) \nleqslant l_1(\beta)\big).$$

We then quasi-order $\mathfrak{L}^B(Q)$ by

$$l_0 \leqslant_{\mathfrak{L}} l_1 \quad \longleftrightarrow \quad \text{Player II has a winning strategy in } G(l_0, l_1).$$

$$
\begin{array}{ccc}
\multicolumn{3}{c}{\overbrace{\mathrm{I} \qquad\quad \mathrm{II}}^{G(l_0, l_1)}} \\
\hline
\alpha_0 & \longrightarrow & \beta_0 \\
 & \swarrow & \\
\alpha_1 & \longrightarrow & \beta_1 \\
 & \swarrow & \\
\alpha_2 & \longrightarrow & \beta_2 \\
 & \swarrow & \\
\vdots & & \vdots \\
\hline
\alpha & & \beta
\end{array}
$$

Figure 5.1: A play of the game $G(l_0, l_1)$.

Remark 5.2. When l_0 and l_1 in $\mathfrak{L}^B(Q)$ have non-empty closed domains, then there are non-empty pruned trees T_0 and T_1 on ω with $\operatorname{dom} l_0 = [T_0]$ and $\operatorname{dom} l_1 = [T_1]$. The game $G(l_0, l_1)$ is then equivalent to the following game where each player is required to stay inside T_0 or T_1 respectively. Player I starts by choosing some $\alpha_0 \in \omega$ with $(\alpha_0) \in T_0$, then Player II chooses some $\beta_0 \in \omega$ with $(\beta_0) \in T_1$, then Player I chooses some $\alpha_1 \in \omega$ such that $(\alpha_0, \alpha_1) \in T_0$, so on and so forth. Both players eventually produce $\alpha \in [T_0]$ and $\beta \in [T_1]$ respectively. Player II wins if and only if $l_0(\alpha) \leqslant l_1(\beta)$ in Q. Clearly the restriction of any winning strategy for II in $G(l_0, l_1)$ to the positions for I which belong to T_0 is a winning strategy for Player II in the above game. Conversely the restriction of any winning strategy for Player I in $G(l_0, l_1)$ to the positions for II which belong to T_1 is a winning strategy for I in the above game.

In particular it is clear from the remark that for every $X, Y \in H^*_{\omega_1}(Q)$ the game $G_{V^*}(X, Y)$ (see Subsection 2.2.4) is equivalent to the game $G(l_X, l_Y)$. Therefore $X \mapsto l_X$ is an embedding of $H^*_{\omega_1}(Q)$ into $\mathfrak{L}^B(Q)$ and we can really see $\mathfrak{L}^B(Q)$ as a Borel generalisation of $H^*_{\omega_1}(Q)$.

In the definition of $\mathfrak{L}^B(Q)$ the restriction to Borel Q-labelling functions on Borel subsets of ω^ω ensures that the game $G(l_0, l_1)$ is Borel and therefore determined by Martin's Borel determinacy [Mar75]. By the axiom of choice, such a restriction is actually necessary to get positive results. However one can weaken this restriction if one is willing to assume the determinacy of a larger class of games.

As observed for the first time by van Engelen, Miller, and Steel [vEMS87, Theorem 3.2] a bad locally constant multi-sequence in $\mathfrak{L}^B(Q)$ can be reflected into Q, in a similar way as we reflected a bad locally constant multi-sequence in $V^*(Q)$ or $H^*_{\omega_1}(Q)$ into Q in Proposition 2.55. But this time we do not get a locally constant multi-sequence into Q in general, but only a Borel multi-sequence.

Definition 5.3. A multi-sequence $f : [X]^\infty \to E$ into some set E is said to be *Borel* if f has a countable image and $f^{-1}(e)$ is Borel in $[X]^\infty$ for every $e \in E$.

The following can be seen as the Borel analogue of Proposition 2.55 and its proof follows the same line (see the discussion in Subsection 2.2.4). This is a slight generalisation of a theorem by van Engelen, Miller, and Steel [vEMS87, Theorem 3.2] (see also the paper by Louveau and Saint Raymond [LS90, Theorem 3]).

Theorem 5.4. *Let Q be a qo. For every bad locally constant multi-sequence $h : [\omega]^\infty \to \mathfrak{L}^B(Q)$ there exists a bad Borel multi-sequence $g : [\omega]^\infty \to Q$ such that $g(X) \in \operatorname{Im}(h(X))$ for every $X \in [\omega]^\infty$.*

Proof. Let $h : [\omega]^\infty \to \mathfrak{L}^B(Q), X \mapsto h_X$ be a bad locally constant multi-sequence. Let L be the countable image of h inside $\mathfrak{L}^B(Q)$ and endow L with the discrete topology. We consider the closed subset of L^ω of 'nowhere ascending' sequences in L:

$$L^{\mathrm{na}} = \{(l_n)_{n\in\omega} \in L^\omega \mid \text{for every } n \in \omega,\ l_n \not\leq_{\mathrm{Lip}} l_{n+1}\}.$$

Notice that L^{na} comes with a shift map $\tilde{\mathsf{S}} : L^{\mathrm{na}} \to L^{\mathrm{na}}$ given by $\tilde{\mathsf{S}}((l_n)_{n\in\omega}) = (l_{n+1})_{n\in\omega}$. We are going to define $g : [\omega]^\infty \to Q$ as the composition of a map $\vec{h} : [\omega]^\infty \to L^{\mathrm{na}}$ with a map $\Psi : L^{\mathrm{na}} \to Q$.

Recall that $\mathsf{S} : [\omega]^\infty \to [\omega]^\infty$ denotes the shift map $X \mapsto {}_*X$ and let $\mathsf{S}^0 = \mathrm{id}_{[\omega]^\infty}$ and $\mathsf{S}^{k+1} = \mathsf{S} \circ \mathsf{S}^k$ for every $k \in \omega$.

Claim. *The map $\vec{h} : [\omega]^\infty \to L^{na}$ defined by $\vec{h}(X) = h(\mathsf{S}^n(X))_{n\in\omega}$ is continuous and we have $\vec{h} \circ \mathsf{S} = \tilde{\mathsf{S}} \circ \vec{h}$.*

Since h is bad, $h(\mathsf{S}^n(X)) \not\leq_{\mathrm{Lip}} h(\mathsf{S}^{n+1}(X))$ for every $n \in \omega$ and so $\vec{h}$ is well defined. If we let $\pi_n : P^\omega \to P$ denote the projection on the n^{th} coordinate, then the map $\pi_n \circ \vec{h} = h(\mathsf{S}^n(X))$ is continuous since S is continuous and h is locally constant. It follows that $\vec{h}$ is continuous and we have proved the Claim.

Claim. *There is a map $\Psi : L^{na} \to Q$ such that*

(i) *for every $\lambda \in L^{na}$ we have $\Psi(\lambda) \not\leq \Psi(\tilde{\mathsf{S}}(\lambda))$ in Q,*

(ii) *$\Psi(\lambda) \in \operatorname{Im} l_0$ for every $\lambda = (l_n)_{n\in\omega} \in L^{na}$,*

(iii) *the image of Ψ is countable and $\Psi^{-1}(q)$ is Borel in L^{na} for every $q \in Q$.*

If $l_0 \not\leq_{\mathfrak{L}} l_1$ in $\mathfrak{L}^B(Q)$, then II has no winning strategy in the Borel game $G(l_0, l_1)$ and therefore Player I has a winning strategy. By the axiom of countable choice we choose for every $l_0, l_1 \in L$ with $l_0 \not\leq_{\mathfrak{L}} l_1$ a winning strategy $\sigma_{l_0}^{l_1}$ for Player I in $G(l_0, l_1)$.

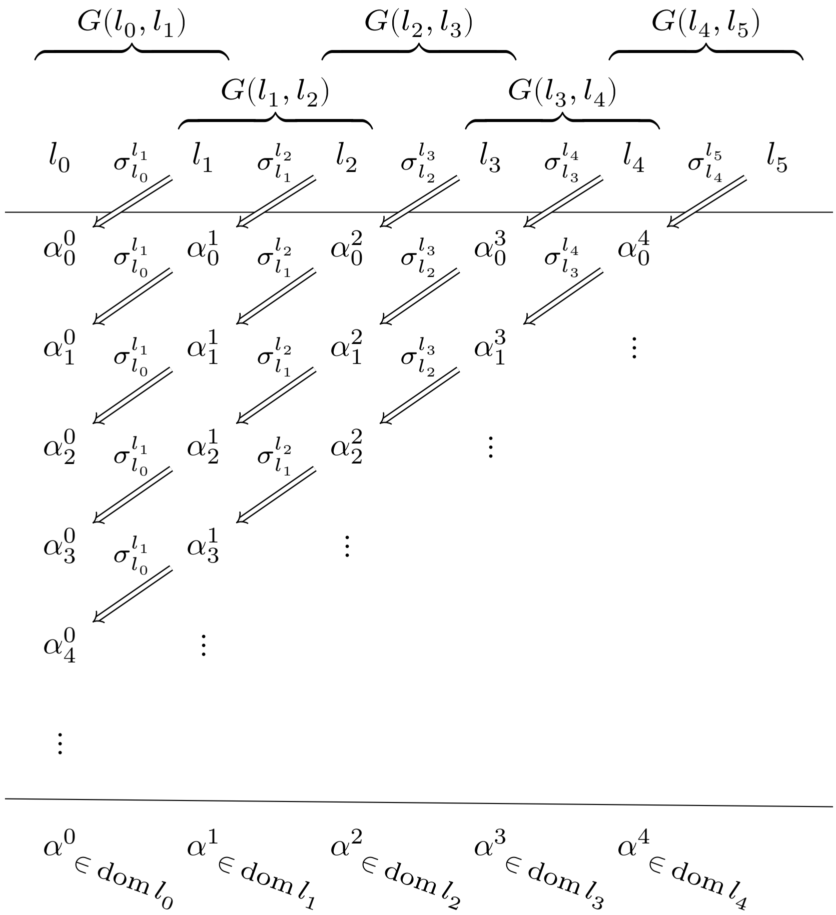


Figure 5.2: Stringing strategies together towards infinity.

For each sequence $\lambda = (l_n)_{n\in\omega} \in L^{\mathrm{na}}$ we define a diagram $(\alpha^i(\lambda))_{i\in\omega} \in \prod_{i\in\omega} \operatorname{dom} l_i$ by stringing strategies together as depicted in Figure 5.2. In each game $G(l_n, l_{n+1})$ Player I follows the strategy $\sigma_{l_n}^{l_{n+1}}$ and Player II copies the moves made by Player I in $G(l_{n+1}, l_{n+2})$. Writing $\alpha^n(\lambda) = (\alpha^n_k)_{k\in\omega}$ this means that for every $n, k \in \omega$ the number α^n_k is the next move for Player I in $G(l_n, l_{n+1})$ provided by the strategy $\sigma_{l_n}^{l_{n+1}}$ in the position where Player II has played $\alpha^{n+1}\restriction_k$. Therefore, for every $n \in \omega$, $(\alpha^n(\lambda), \alpha^{n+1}(\lambda))$ is a play of $G(l_n, l_{n+1})$ in which Player I has followed the winning strategy $\sigma_{l_n}^{l_{n+1}}$ and so necessarily $\alpha^n(\lambda) \in \operatorname{dom} l_n$. It follows that for every $n \in \omega$ we have $l_n(\alpha^n(\lambda)) \not\leq l_{n+1}(\alpha^{n+1}(\lambda))$.

We define $\Psi : L^{\mathrm{na}} \to Q$ by $\Psi(\lambda) = l_0(\alpha^0(\lambda))$ for every $\lambda = (l_n)_{n\in\omega} \in L^{\mathrm{na}}$.

We have $\Psi((l_n)_{n\in\omega}) \in \operatorname{Im} l_0$ for every $(l_n)_{n\in\omega} \in L^{\mathrm{na}}$ and therefore the image of Ψ is contained in the countable union of countable sets $\bigcup_{l\in L} \operatorname{Im} l$. For $l \in L$, let us denote by $L^{\mathrm{na}}_l = \{(l_n)_n \in L^{\mathrm{na}} \mid l_0 = l\}$ the clopen subset of L^{na} of sequences starting with l. For every $l \in L$, the map $\alpha^0\restriction_{L^{\mathrm{na}}_l} : L^{\mathrm{na}}_l \to \operatorname{dom} l$ is continuous since for every $\lambda = (l_n)_n \in L^{\mathrm{na}}_l$ the initial segment $\alpha^0(\lambda)\restriction_k$ depends only on $l_1, \dots, l_k$ – and it is determined by $\sigma_{l_0}^{l_1}, \sigma_{l_1}^{l_2} \dots \sigma_{l_{k-1}}^{l_k}$ – as easily seen by contemplating Figure 5.2. Therefore for every $q \in Q$,

$$\Psi^{-1}(q) = \bigcup_{l\in L} \left(l \circ \alpha^0\restriction_{L^{\mathrm{na}}_l}\right)^{-1}(q)$$

is Borel since L is countable, l is Borel and $\alpha^0\restriction_{L^{\mathrm{na}}_l}$ is continuous.

Moreover, by construction for every $\lambda = (l_n)_{n\in\omega} \in L^{\mathrm{na}}$, we have $\alpha^1(\lambda) = \alpha^0(\tilde{\mathsf{S}}(\lambda))$ and therefore

$$\Psi(\lambda) = l_0(\alpha^0(\lambda)) \not\leq l_1(\alpha^1(\lambda)) = l_1(\alpha^0(\tilde{\mathsf{S}}(\lambda))) = \Psi(\tilde{\mathsf{S}}(\lambda)),$$

as $\sigma_{l_0}^{l_1}$ is winning for Player I in $G(l_0, l_1)$. This finishes the proof of the second Claim.

Finally we define the multi-sequence $g : [\omega]^\infty \to Q$ by letting $g = \Psi \circ \vec{h}$. We have for every $X \in [\omega]^\infty$

$$g(X) = \Psi(\vec{h}(X)) \not\leq \Psi(\tilde{\mathsf{S}}(\vec{h}(X))) = \Psi(\vec{h}(\mathsf{S}(X))),$$

hence g is bad. Moreover g is a Borel multi-sequence since for every $q \in Q$, the set

$$g^{-1}(q) = \vec{h}^{-1}(\Psi^{-1}(q))$$

is Borel in $[\omega]^\infty$ since $\Psi^{-1}(q)$ is Borel in L^{na} and $\vec{h}$ preserves Borel sets by preimages since it is continuous. □

The previous theorem only yields a Borel multi-sequence and we are therefore led to consider the combinatorial properties of Borel subsets of $[\omega]^\infty$. Notice that Nash-Williams Theorem 2.36 is equivalent to the statement that for every clopen subset C of $[\omega]^\infty$ there exists $X \in [\omega]^\infty$ such that either $[X]^\infty \subseteq C$ or $[X]^\infty \cap C = \emptyset$. Galvin and Prikry [GP73] proved the following important generalisation:

Theorem 5.5 (Galvin and Prikry)**.** *For every Borel subset B of $[\omega]^\infty$ there exists $X \in [\omega]^\infty$ such that either $[X]^\infty \subseteq B$ or $[X]^\infty \cap B = \emptyset$.*

The proof of this theorem uses an important technique sometimes called combinatorial forcing. Several textbooks (notably [Tod10; Kec95; Hal11; Jec03]) provide a proof of this theorem and its generalisation to the sets enjoying the Baire property relatively to the Ellentuck topology. In our context, we are interested in the following well-known corollary (see also [PV92, Theorem 3.4], [LS82]).

Theorem 5.6. *Let E be a set. Every Borel multi-sequence into E admits a locally constant sub-multi-sequence. In particular a quasi-order Q is* BQO *if and only if there is no bad Borel multi-sequence into Q.*

Proof (Folklore). Let $g : [\omega]^\infty \to Q$ be a Borel multi-sequence and let $(B^i)_{i\in\omega}$ enumerate the countable Borel partition $\{f^{-1}(q) \mid q \in \operatorname{Im} g\}$ of $[\omega]^\infty$. We show that there exists $X \in [\omega]^\infty$ such that for each $B_k \cap [X]^\infty$ is clopen in $[X]^\infty$, therefore showing that the sub-multi-sequence $g : [X]^\infty \to Q$ is locally constant. To do so, we define a sequence $(N_i)_{i\in\omega} \subseteq [\omega]^\infty$ with $n_i = \min N_i$ such that for every $i \in \omega$:

(i) $N_{i+1} \in [_*N_i]^\infty$,

(ii) for every $s \subseteq \{n_j \mid j < i\}$

$$\text{either } \{s \cup Z \mid Z \in [N_i]^\infty\} \subseteq B^i, \text{ or } \{s \cup Z \mid Z \in [N_i]^\infty\} \cap B^i = \emptyset,$$

By Theorem 5.5, there exists $N_0 \in [\omega]^\infty$ such that either $[N_0]^\infty \subseteq B_0$ or $B_0 \cap [N_0]^\infty = \emptyset$. Now assume the sequence $(N_i)_i$ is defined up to some $i \geqslant 0$. For each $s \subseteq \{n_j \mid j < i+1\}$ we consider the following Borel subset of $[_*N_i]^\infty$:

$$B^{i+1}_s = \{Z \in [_*N_i]^\infty \mid s \cup Z \in B^{i+1}\}.$$

Let s_l for $l < 2^{i+1}$ enumerate the subsets of $\{n_j \mid j < i+1\}$. Applying Theorem 5.5 to $B^{i+1}_{s_0}$ we find $Y_0 \in [_*N_i]^\infty$ such that either $[Y_0]^\infty \subseteq B^{i+1}_{s_0}$ or

$[Y_0]^\infty \cap B^{i+1}_{s_0} = \emptyset$. Next we apply Theorem 5.5 to $B^{i+1}_{s_1}$ to find $Y_1 \in [Y_0]^\infty$ such that either $[Y_0]^\infty \subseteq B^{i+1}_{s_1}$ or $[Y_0]^\infty \cap B^{i+1}_{s_1} = \emptyset$. We go on like this for 2^{i+1} steps and we let $N_{i+1} = Y_{2^{i+1}}$ which satisfies the requirements.

Finally let $X = \{n_i \mid i \in \omega\}$. For any $i \in \omega$ let S_i denote the finite set of those $s \subseteq \{n_j \mid j < i\}$ such that $\{s \cup Z \mid Z \in [N_i]^\infty\} \subseteq B^i$. Then for every $i \in \omega$ we have

$$B^i \cap [X]^\infty = \bigcup_{s \in S_i} M_s \cap [X]^\infty,$$

where we recall that $M_s = \{Y \in [\omega]^\infty \mid s \sqsubset Y\}$ is a basic clopen set. Therefore each $B^i \cap [X]^\infty$ is clopen in $[X]^\infty$ as desired. $\square$

Therefore we have obtained the following:

Theorem 5.7. *Let Q be a quasi-order. If Q is* BQO*, then $\mathfrak{L}^B(Q)$ is* BQO*.*

Remark that since $H^*_{\omega_1}(Q)$ embeds into $\mathfrak{L}^B(Q)$ via $X \mapsto l_X$, it follows from Theorem 2.59 that if $\mathfrak{L}^B(Q)$ is well-founded (in particular if it is WQO), then Q is BQO. Hence in a sense a quasi-order Q relates to $\mathfrak{L}^B(Q)$ in the same way as it does to $H^*_{\omega_1}(Q)$.

In the game $G(l_0, l_1)$, any winning strategy σ for Player II induces continuous maps $\sigma^* : \omega^\omega \to \omega^\omega$ such that for every $\alpha \in \operatorname{dom} l_0$ we have $\sigma^*(\alpha) \in \operatorname{dom} l_1$ and $l_0(\alpha) \leqslant l_1(\sigma^*(\alpha))$. In particular the function σ^* restricts to a continuous map $f : \operatorname{dom} l_0 \to \operatorname{dom} l_1$ such that for every $\alpha \in \operatorname{dom} l_0$ we have $l_0(\alpha) \leqslant l_1(f(\alpha))$. This allows us to bring $\mathfrak{L}^B(Q)$ to the topological setting.

A Luzin space – or Borel absolute space – is a separable metrisable space which is Borel in every Polish space in which it embeds. It is well-known a space $\mathcal{X}$ is Luzin if and only if there exists a continuous bijection $f : F \to \mathcal{X}$ from a closed subset F of ω^ω, if and only if it is homeomorphic to a Borel subspace of some Polish space (see for example the article by Dellacherie [Del80, p.208]).

Since every separable metrisable zero-dimensional space – or equivalently every zero-dimensional second countable space – is homeomorphic to a subspace of the Baire space, the zero-dimensional Luzin spaces are the topological spaces which are homeomorphic to a Borel subspace of ω^ω.

Definition 5.8. Let Q be a quasi-order. A map $l : \mathcal{X} \to Q$ from some Luzin zero-dimensional space $\mathcal{X}$ is called a *Borel Q-labelling function of $\mathcal{X}$* if

(1) for every $q \in Q$, the set $l^{-1}(q)$ is Borel in $\mathcal{X}$,

(2) the image $\operatorname{Im} l$ of l is countable.

We refer to $\mathcal{X}$ as the domain of l and denote it by $\operatorname{dom} l$. We let $B(Q)$ be the class of all Borel Q-labelling of some zero-dimensional Luzin space.

We quasi-order $B(Q)$ by

$$l_0 \leqslant_{\mathrm{c}} l_1 \quad\longleftrightarrow\quad \begin{array}{l}\text{there exists a continuous map } g : \operatorname{dom} l_0 \to \operatorname{dom} l_1 \text{ s.t.}\\ \text{for every } x \in \operatorname{dom} l_0 \text{ we have } l_0(x) \leqslant l_1(g(x)) \text{ in } Q.\end{array}$$

Theorem 5.9. *If Q is* BQO*, then $B(Q)$ is* BQO*.*

Proof. Let $h : [\omega]^\infty \to B(Q)$ be a locally constant multi-sequence. Since h has a countable image and every zero-dimensional Luzin space is homeomorphic to a Borel subset of ω^ω, we can assume that in fact $h : [\omega]^\infty \to \mathfrak{L}^B(Q)$. By Theorem 5.7, if Q is BQO then there exist $X \in [\omega]^\infty$ such that $h(X) \leqslant_{\mathfrak{L}} h({}_*X)$, i.e. Player II has a winning strategy in $G(h(X), h({}_*X)$. Now a winning strategy σ for Player II in $G(h(X), h({}_*X)$ induces a continuous map $\sigma^* : \omega^\omega \to \omega^\omega$ which restricts to a continuous $f : \operatorname{dom} h(X) \to \operatorname{dom} h({}_*X)$ such that $h(X)(\alpha) \leqslant h({}_*X)(f(\alpha))$ for every $\alpha \in \operatorname{dom} h(X)$. Therefore $h(X) \leqslant_{\mathrm{c}} h({}_*X)$ and $h : [\omega]^\infty \to B(Q)$ is good. Since h was arbitrary, it follows that $B(Q)$ is BQO. □

The case of the BQO $2 = \{0, 1\}$ partially ordered by equality has received much attention. In this case, $B(2)$ consists of the characteristic functions of Borel subsets of zero-dimensional Luzin spaces quasi-ordered by continuous reducibility, and by Theorem 5.9 $B(2)$ is BQO. Let us write any $l \in B(2)$ as the ordered pair $(A_l, \operatorname{dom} l)$ where $A_l = l^{-1}(1)$. Conversely we can write every Borel subset A of some zero-dimensional Luzin space $\mathcal{X}$ as the element of $B(Q)$ consisting in its characteristic function $l_A : \mathcal{X} \to 2$. Now for every $l_0, l_1 \in B(Q)$ we have $l_0 \leqslant_{\mathrm{c}} l_1$ if and only if there exists a continuous function $f : \operatorname{dom} l_0 \to \operatorname{dom} l_1$ such that $A_{l_0} = f^{-1}(A_{l_1})$. We therefore also write $(A, \mathcal{X}) \leqslant_{\mathrm{W}} (B, \mathcal{Y})$ in place of $l_A \leqslant_{\mathrm{c}} l_B$.

The symmetry of the qo 2 allows us to prove the following result, usually referred to as the ‘Wadge Lemma’.

Lemma 5.10 (Wadge Lemma)**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be zero-dimensional Luzin spaces, $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$ be Borel. Then either $(A, \mathcal{X}) \leqslant_W (B, \mathcal{Y})$ or $(B, \mathcal{Y}) \leqslant_W (\mathcal{X} \setminus A, \mathcal{X})$.*

Proof. We can assume that $\mathcal{X}$ and $\mathcal{Y}$ are Borel subsets D_0 and D_1 of ω^ω. Let $l_A : D_0 \to 2$ and $l_B : D_1 \to 2$ be the characteristic functions of A and B respectively. If $(A, \mathcal{X}) \leqslant_{\mathrm{W}} (B, \mathcal{Y})$ then Player II has no winning strategy in the Borel game $G(l_A, l_B)$ and so Player I has a winning strategy. Notice that any winning strategy τ for Player I in $G(l_A, l_B)$ induces a continuous map

$\tau^* : \omega^\omega \to \omega^\omega$ with the property that $\tau^*(\beta) \in D_0$ for every $\beta \in \omega^\omega$ and if $\beta \in D_1$, then $l_A(f(\beta)) \neq l_B(\beta)$. Therefore when we restrict τ^* to D_1 we get a continuous map $f : D_1 \to D_0$ such that

$$\text{for every } \beta \in D_1 \quad (f(\beta) \notin A \quad \longleftrightarrow \quad \beta \in B),$$

and therefore $(B, \mathcal{Y}) \leqslant_{\mathrm{W}} (\mathcal{X} \setminus A, \mathcal{X})$ as desired. $\square$

One consequence of Wadge Lemma is that antichains in $B(2)$ are of size at most two. To see this, observe that $(A, \mathcal{X}) \leqslant_{\mathrm{W}} (B, \mathcal{Y})$ if and only if $(A^{\complement}, \mathcal{X}) \leqslant_{\mathrm{W}} (B^{\complement}, \mathcal{Y})$. Suppose that $(A, \mathcal{X})$ and $(B, \mathcal{Y})$ are incomparable. It follows from Wadge Lemma that that $(B, \mathcal{Y})$ is equivalent to $(A^{\complement}, \mathcal{X})$, and therefore $(A, \mathcal{X})$ is equivalent to $(B^{\complement}, \mathcal{Y})$. Let now $(C, \mathcal{Z})$ be an arbitrary element of $B(2)$. Then by Wadge Lemma again, we have either $(A, \mathcal{X}) \leqslant_{\mathrm{W}} (C, \mathcal{Z})$ or $(C, \mathcal{Z}) \leqslant_{\mathrm{W}} (A^{\complement}, \mathcal{X})$.

$$\begin{array}{ccccc}
 & & (C, \mathcal{Z}) & & \\
 & \not\leqslant_{\mathrm{W}} & & & \\
(A, \mathcal{X}) & |_{\mathrm{W}} & & (B, \mathcal{Y}) & \equiv_{\mathrm{W}} (A^{\complement}, \mathcal{X}) \\
 & & & \not\leqslant_{\mathrm{W}} & \\
 & & (C, \mathcal{Z}) & &
\end{array}$$

Figure 5.3: Consequences of Wadge Lemma

Therefore antichains are of size at most two, and up to equivalence they are necessarily of the form $(A, \mathcal{X})$ and $(A^{\complement}, \mathcal{X})$. As we have just seen a second consequence of Wadge Lemma is that if we stipulate that $(A, \mathcal{X})$ and $(A^{\complement}, \mathcal{X})$ are always equivalent, this makes $B(2)$ into a linear order, and therefore a well-order.

While the continuous reducibility is ideal in the zero-dimensional framework, the situation is dramatically different in spaces like the real line for example. As far as we know the only way to prove that the continuous reducibility is well-founded on Borel subsets of the Baire space relies on two player games with perfect information. Moreover we think that the notion of dimension zero in the topological setting compares with the idea of perfect information in the game-theoretic setting. This may indicate that the restriction to zero-dimensional spaces in Theorem 5.9 is in a sense necessary.

We aim to generalise the continuous reducibility – and in fact Theorem 5.9 – to spaces of higher dimension while maintaining the nice properties it enjoys on zero-dimensional spaces. Our first step in this direction is very simple but fundamental. We observe that the notion of function is not essential to the idea of reduction, but that on the contrary one can advantageously consider total relations in place of functions.

5.2 Reductions as total relations

The concept of reduction is used in several different fields, such as complexity theory, automata theory and descriptive set theory. While particular definitions relies on different concepts, they all share a general idea. If $\mathcal{X}$ and $\mathcal{Y}$ are sets, $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$, a function $f : \mathcal{X} \to \mathcal{Y}$ is called a *reduction of A to B* if $A = f^{-1}(B)$ or equivalently if

$$\forall x \in \mathcal{X} \quad (x \in A \quad \longleftrightarrow \quad f(x) \in B)\,.$$

Let $\mathcal{F}$ be a class of functions from $\mathcal{X}$ to $\mathcal{X}$ that contains the identity on $\mathcal{X}$ and that is closed under composition. For $A, B \subseteq \mathcal{X}$ we say that A is *reducible* to B with respect to $\mathcal{F}$ if there exists $f \in \mathcal{F}$ such that f is a reduction of A to B. This always defines a quasi-order on the powerset of $\mathcal{X}$.

We now observe that as far as reducibility is concerned reductions do not need to be functions. In fact one may as well consider *total relations* in place of functions.

We say $R \subseteq \mathcal{X} \times \mathcal{Y}$ is a *(total) relation from $\mathcal{X}$ to $\mathcal{Y}$*, in symbols $R : \mathcal{X} \rightrightarrows \mathcal{Y}$, if for all $x \in \mathcal{X}$ there exists $y \in \mathcal{Y}$ with $(x, y) \in R$. We also write $R(x, y)$ in place of $(x, y) \in R$.

Definition 5.11. If $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$ we say that a *reduction of A to B* is a total relation $R : \mathcal{X} \rightrightarrows \mathcal{Y}$ such that

$$\forall x \in \mathcal{X}\ \forall y \in \mathcal{Y}\ \Big[R(x, y) \to (x \in A \leftrightarrow y \in B)\Big]. \tag{5.1}$$

One can also view a relation $R \subseteq \mathcal{X} \times \mathcal{Y}$ as the function

$$\begin{aligned} R^{\rightarrow} : \mathcal{X} &\longrightarrow \mathcal{P}(\mathcal{Y}) \\ x &\longmapsto R^{\rightarrow}(x) = \{y \in \mathcal{Y} \mid R(x, y)\}. \end{aligned}$$

From this point of view, R is total from $\mathcal{X}$ to $\mathcal{Y}$ if and only if $R^{\rightarrow}(x) \neq \emptyset$ for all $x \in \mathcal{X}$, and (5.1) can be stated in the following way:

$$\forall x \in \mathcal{X}\ \Big[\big(x \in A \wedge R^{\rightarrow}(x) \subseteq B\big) \vee \big(x \in A^{\complement} \wedge R^{\rightarrow}(x) \subseteq B^{\complement}\big)\Big]. \tag{5.2}$$

Of course, for every function $f : \mathcal{X} \to \mathcal{Y}$, f is a reduction of A to B if and only if its graph $\{(x, f(x)) \mid x \in \mathcal{X}\}$, as a total relation from $\mathcal{X}$ to $\mathcal{Y}$, is a reduction of A to B. So our notion of reduction as total relations subsumes the notion of reduction as functions.

Observe also that it follows directly from (5.2) that a total relation R is a reduction of A to B if and only if it is a reduction from $A^{\complement}$ to $B^{\complement}$.

Two total relations $R : \mathcal{X} \rightrightarrows \mathcal{Y}$ and $S : \mathcal{Y} \rightrightarrows \mathcal{Z}$ compose to yield the total relation $S \circ R : \mathcal{X} \rightrightarrows \mathcal{Z}$ in the expected way

$$S \circ R = \{(x, z) \in \mathcal{X} \times \mathcal{Z} \mid \exists y \in \mathcal{Y}\ R(x, y) \wedge S(y, z)\}.$$

Fact 5.12. *If $A \subseteq \mathcal{X}$, $B \subseteq \mathcal{Y}$, $C \subseteq \mathcal{Z}$, $R : \mathcal{X} \rightrightarrows \mathcal{Z}$ is a reduction of A to B and $S : \mathcal{Y} \rightrightarrows \mathcal{Z}$ is a reduction of B to C, then $S \circ R : \mathcal{X} \rightrightarrows \mathcal{Z}$ is a reduction of A to C.*

Let $\mathcal{R}$ be a class of total relations from $\mathcal{X}$ to $\mathcal{X}$ that contains the diagonal $\{(x, x) \mid x \in \mathcal{X}\}$ and that is closed under composition. For $A, B \subseteq \mathcal{X}$ we say that A is reducible to B with respect to $\mathcal{R}$ if there $R \in \mathcal{R}$ such that R is a reduction of A to B. Again this defines a quasi-order on the powerset of $\mathcal{X}$ that we call $\mathcal{R}$-reducibility.

The following fact follows immediately from (5.1).

Fact 5.13. *Let $R, S : \mathcal{X} \rightrightarrows \mathcal{Y}$, $A \subseteq \mathcal{X}$, $B \subseteq \mathcal{Y}$. If $R \subseteq S$ and S is a reduction of A to B, then R is also a reduction of A to B.*

Consequently, for a class $\mathcal{R}$ as above, if we consider the the upward closure of $\mathcal{R}$ defined by

$$\overline{\mathcal{R}} = \{S : \mathcal{X} \rightrightarrows \mathcal{X} \mid \exists R \in \mathcal{R}\ R \subseteq S\},$$

then the $\mathcal{R}$-reducibility equals the $\overline{\mathcal{R}}$-reducibility. Therefore as far as reducibility is concerned, we gain generality by considering classes of total relations instead of classes of functions, and we can always consider classes of total relations that are upward closed.

In the next section the definition of a notion of continuity for total relations between second countable T_0 spaces is provided. We then discuss in the following sections some properties of the reducibility associated with this class of 'continuous' total relations.

5.3 Relatively continuous relations

We are interested in a notion of continuity for total relations called *relative continuity*. It relies on the concept of *admissible representation of a topological* *space*. While this concept is fundamental to Type-2 Theory of Effectivity (see the textbook by Weihrauch [Wei00]), we do not expect our reader to be familiar with the simple and interesting underpinning of this approach to computable analysis.

We therefore review the basic definitions and provide proofs for the convenience of the reader.

5.3.1 Admissible representations

A topological space $\mathcal{X}$ is called *second countable* if it admits a countable base of open sets. It satisfies the separation axiom T_0 if every two distinct points are topologically distinguishable, i.e. for any two distinct points x and y there is an open set which contains one of these points and not the other. It is called zero-dimensional, or 0-*dimensional*, if it admits a base of clopen sets, i.e. of simultaneously open and closed sets. A space is second countable and 0-dimensional if and only if it is homeomorphic to a subset of ω^ω. A Polish space is a second countable completely metrisable topological space, the Baire space is a crucial example of Polish space. Recall [Kec95, (3.11), p.17] that a subspace of a Polish space is Polish if and only if it $\mathbf{\Pi}^0_2$, i.e. a countable intersection of open sets.

Let $\mathcal{X}, \mathcal{Y}$ be second countable T_0 spaces. If $A \subseteq \mathcal{X}$ and $f : A \to \mathcal{Y}$ is a function, f is called a *partial function* from $\mathcal{X}$ to $\mathcal{Y}$, in symbols $f :\subseteq \mathcal{X} \to \mathcal{Y}$, and we refer to A as the domain of f, denoted by $\operatorname{dom} f$. A partial function $f :\subseteq \mathcal{X} \to \mathcal{Y}$ is continuous if it is continuous on its domain for the subspace topology on $\operatorname{dom} f$, i.e. if for every open U of $\mathcal{Y}$ there is an open V of $\mathcal{X}$ such that $f^{-1}(U) = V \cap \operatorname{dom} f$.

We quasi-order the partial functions from ω^ω into $\mathcal{X}$ by saying that for $f, g :\subseteq \omega^\omega \to \mathcal{X}$

$$f \leqslant^{\text{rep}} g \quad \longleftrightarrow \quad \begin{cases} \text{there exists a continuous } h : \operatorname{dom} f \to \operatorname{dom} g \\ \text{with } f(\alpha) = g \circ h(\alpha) \text{ for all } \alpha \in \operatorname{dom} f. \end{cases}$$

Clearly, if g is continuous and $f \leqslant^{\text{rep}} g$, then f is continuous too. Hence the set of partial continuous functions from ω^ω into $\mathcal{X}$ is downward closed with respect to $\leqslant^{\text{rep}}$.

Definition 5.14 ([Wei00])**.** Let $\mathcal{X}$ be a topological space. A partial continuous function $\rho :\subseteq \omega^\omega \to \mathcal{X}$ is called an *admissible representation* of $\mathcal{X}$ if it is a $\leqslant^{\text{rep}}$-greatest element among partial continuous functions to $\mathcal{X}$, i.e. $f \leqslant^{\text{rep}} \rho$ holds for every partial continuous $f :\subseteq \omega^\omega \to \mathcal{X}$.

Observe that an admissible representation ρ of $\mathcal{X}$ is necessarily onto $\mathcal{X}$, since for every point $x \in \mathcal{X}$, we have $c_x \leqslant^{\text{rep}} \rho$ where $c_x : \omega^\omega \to \mathcal{X}$, $\alpha \mapsto x$ is the constant function.

Remark 5.15. Since the subspaces of ω^ω are up to homeomorphism the second countable 0-dimensional spaces, an admissible representation of $\mathcal{X}$ is also a continuous map $\rho : D \to \mathcal{X}$ from some second countable 0-dimensional space D such that for every continuous map $g : E \to \mathcal{X}$ from a second countable 0-dimensional space E there exists a continuous map $h : E \to D$ such that $g = \rho \circ h$.

Now it is well known that every second countable T_0 space $\mathcal{X}$ has an admissible representation. Since it is simple and crucial for the sequel we now explain this fact in detail.

Definition 5.16. Let $\mathcal{X}$ be a second countable T_0 space and $(V_n)_{n\in\omega}$ be a countable base of open sets for $\mathcal{X}$. We define the *standard representation* of $\mathcal{X}$ with respect to (V_n) to be the partial map $\rho :\subseteq \omega^\omega \to \mathcal{X}$ defined by

$$\rho(\alpha) = x \quad \longleftrightarrow \quad \{n \mid \exists k\ \alpha(k) = n+1\} = \{n \mid x \in V_n\}.$$

Notice that in the definition ρ is indeed a function on its domain because $\mathcal{X}$ is T_0. An $\alpha \in \omega^\omega$ codes via ρ a point $x \in \mathcal{X}$ if and only if α enumerates the indices of all the V_n's to which x belongs, while 0 can be thought of as an index for the whole space $\mathcal{X}$ – which may not appear among the V_n's.

Theorem 5.17. *For every second countable T_0 space $\mathcal{X}$ there exists an admissible representation $\rho :\subseteq \omega^\omega \to \mathcal{X}$. Moreover it can be chosen such that*

(i) *ρ is open,*

(ii) *for every $x \in \mathcal{X}$, the fibre $\rho^{-1}(x)$ is Polish.*

Proof. Let (V_n) be a countable base for $\mathcal{X}$ and let $\rho :\subseteq \omega^\omega \to \mathcal{X}$ be the standard representation of $\mathcal{X}$ with respect to (V_n). It is enough to show that ρ satisfies all the requirements.

Continuity: Note that $\rho^{-1}(V_n) = \{\alpha \in \operatorname{dom}\rho \mid \exists k\ \alpha(k) = n+1\}$ is open in ω^ω for every n, so ρ is continuous.

Openness: For every basic $N_s = \{x \in \omega^\omega \mid s \subseteq x\}$, $s \in \omega^{<\omega}$, we have $\rho(N_s) = \bigcap_{k<|s|} V_{s_k - 1}$ (where $V_{-1} = \mathcal{X}$ by convention) which is open in $\mathcal{X}$, so ρ is an open map.

Polish fibres: For every point $x \in \mathcal{X}$

$$\rho(\alpha) = x \quad \longleftrightarrow \quad \forall n\big[(\exists k\ \alpha(k) = n+1) \leftrightarrow x \in V_n\big]$$

is a $\mathbf{\Pi}^0_2$ definition of the fibre in x, so ρ has Polish fibres.

Admissibility: Let $f :\subseteq \omega^\omega \to \mathcal{X}$ be continuous. Note that for every $n \in \omega$ and every $\alpha \in \operatorname{dom} f$

$$\exists k\ f(N_{\alpha\restriction_k}) \subseteq V_n \quad \longleftrightarrow \quad f(\alpha) \in V_n.$$

Let $k \mapsto ((k)_0, (k)_1)$ be some bijection from ω to $\omega \times \omega$ whose inverse is denoted by $\langle (k)_0, (k)_1 \rangle = k$. We define $h : \operatorname{dom} f \to \operatorname{dom} \rho$ by

$$h(\alpha)(k) = \begin{cases} (k)_1 + 1 & \text{if } f(N_{\alpha\restriction_{(k)_0}}) \subseteq V_{(k)_1}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly h is continuous. Moreover for every $\alpha \in \operatorname{dom} f$ and every $n \in \omega$, if $f(\alpha) \in V_n$ then there exists $k \in \omega$ such that $f(N_{\alpha\restriction_k}) \subseteq V_n$ and so $h(\alpha)(\langle k, n \rangle) = n + 1$.

Conversely, if for $\alpha \in \operatorname{dom} f$ we have $h(\alpha)(k) = n+1$ then it means that $f(N_{\alpha\restriction_{(k)_0}}) \subseteq V_n$ and so $f(\alpha) \in V_n$.

It follows that $f(\alpha) = \rho \circ h(\alpha)$ for all $\alpha \in \operatorname{dom} f$, as desired. $\square$

Remark 5.18. *i.* If (V_n) is a countable base of a T_0 space $\mathcal{X}$, the map $\sigma :\subseteq \omega^\omega \to \mathcal{X}$ defined by

$$\sigma(\alpha) = x \quad \longleftrightarrow \quad \operatorname{Im} \alpha = \{n \mid x \in V_n\}.$$

is also an admissible representation of $\mathcal{X}$. Indeed σ is continuous and for ρ the standard representation of $\mathcal{X}$ with respect to (V_n) we have $\rho \leqslant^{\mathrm{rep}} \sigma$. To see this we define a monotone map $\varphi : \omega^{<\omega} \to \omega^{<\omega}$ by induction on the length by:

$$\begin{aligned} \varphi(\emptyset) =& \emptyset \\ \varphi(s \frown 0) =& \begin{cases} \varphi(s) \frown \varphi(s)(0) & \text{if } \varphi(s) \neq \emptyset \\ \emptyset & \text{otherwise,} \end{cases} \\ \varphi(s \frown (n+1)) =& \varphi(s) \frown n. \end{aligned}$$

This map φ induces a continuous map $\varphi^* : \operatorname{dom} \varphi^* \to \omega^\omega$ where $\varphi^*(\alpha) = \bigcup_{n\in\omega} \varphi(\alpha\restriction_n)$ for $\alpha \in \{\alpha \in \omega^\omega \mid \lim_{n\to\infty} \varphi(\alpha\restriction_n) = \infty\} = \operatorname{dom} \varphi^*$([Kec95, (2.6), p.8]). We have $\operatorname{dom}\rho \subseteq \operatorname{dom}\varphi^*$ for if $\alpha \in \operatorname{dom}\rho$ then there exists n with $\rho(\alpha) \in V_n$, so there exists a k minimal such that $\alpha(k) > 0$. Hence if $\alpha = 0^k \frown \alpha(k) \frown \beta$ then $\varphi^*(\alpha)(n) = \alpha(n+k) - 1$ when $\alpha(n+k) > 0$ and $\varphi^*(\alpha)(n) = \alpha(k) - 1$ if $\alpha(n+k) = 0$. Moreover we clearly have $= \rho = \sigma \circ \varphi^*$.

ii. However, a standard representation ρ of a second countable space $\mathcal{X}$ relatively to some base has the following advantage. If one chooses the bijection $k \mapsto ((k)_0, (k)_1)$ from ω to $\omega \times \omega$ such that $(k)_0 \leqslant k$ for every k, then the map $h : \operatorname{dom} f \to \operatorname{dom}\rho$ defined in the proof of Theorem 5.17 as a witness to the fact that $f \leqslant^{\mathrm{rep}} \rho$ is actually Lipschitz, i.e. $h(\alpha)\restriction_n$ depends only on $\alpha\restriction_n$ for every $\alpha \in \operatorname{dom} f$ and every $n \in \omega$. This is sometimes convenient.

Importantly, Brattka [Bra99, Corollary 4.4.12] showed that every Polish space X has a *total admissible representation*, i.e. an admissible representation $\rho :\subseteq \omega^\omega \to X$ with $\operatorname{dom}\rho = \omega^\omega$. As an easy consequence one gets that for every second countable T_0 space X: there exists an admissible representation of X with a Polish domain if and only if there exists a total admissible representation of X. Motivated by the rich theory of Polish spaces, it is natural to consider the class of those second countable T_0 spaces which have a total admissible representation. As a matter of fact de Brecht [deB13] showed that this class coincides with the class of *quasi-Polish* spaces that he recently introduced. Moreover he showed that many classical results of descriptive set theory can be generalised to this large class of non necessarily Hausdorff spaces and that the metrisable quasi-Polish spaces are exactly the Polish spaces.

The real line $\mathbb{R}$ is certainly the most important example of non zero-dimensional Polish space and we now introduce two different admissible representations for it.

Example 5.19. Let $(q_n)_{n\in\omega}$ be an enumeration of the rationals and let $I_n = (q_{n_0}, q_{n_1})$ be an enumeration of the non empty intervals of the real line $\mathbb{R}$ with rational endpoints.

We define $\rho_{\mathbb{R}} :\subseteq \omega^\omega \to \mathbb{R}$ relatively to the enumerated base $(I_n)_{n\in\omega}$ by

$$\rho_{\mathbb{R}}(\alpha) = x \quad \longleftrightarrow \quad \operatorname{Im}\alpha = \{n \mid x \in I_n\},$$

so that $\alpha \in \omega^\omega$ codes $x \in \mathbb{R}$ if and only if α enumerates all the intervals with rational endpoints in which x belongs.

The second admissible representation is based on Cauchy sequences and it works mutatis mutandis for every separable complete metric space.

Example 5.20. Let $(q_n)_{n\in\omega}$ be an enumeration of the rationals, and let d be the euclidean metric on $\mathbb{R}$. A sequence $(x_k)_{k\in\omega}$ is said to be *rapidly Cauchy* if for every $i, j \in \omega$, $i < j$ implies $d(x_i, x_j) \leqslant 2^{-i}$. The Cauchy representation $\sigma_{\mathbb{R}} :\subseteq \omega^\omega \to \mathbb{R}$ of the real line is defined by

$$\sigma_{\mathbb{R}}(\alpha) = x \quad \longleftrightarrow \quad \left(q_{\alpha(k)}\right)_{k\in\omega} \text{ is rapidly Cauchy and } \lim_{k\to\infty} q_{\alpha(k)} = x.$$

This is an admissible representation of $\mathbb{R}$.

To illustrate our ideas in the non-metrisable case we consider the Scott Domain $\mathcal{P}\omega$, namely the powerset of ω partially ordered by inclusion and endowed with the Scott topology. A base of $\mathcal{P}\omega$ is given by sets of the form $O_F = \{X \subseteq \omega \mid F \subseteq X\}$ for some finite $F \subseteq \omega$. This space is universal among the second countable T_0 spaces. Indeed for every T_0 space $\mathcal{X}$ with some countable base $(V_n)_{n\in\omega}$ the map $e : \mathcal{X} \to \mathcal{P}\omega$, $x \mapsto \{n \mid x \in V_n\}$ is an embedding.

Example 5.21. The enumeration representation of $\mathcal{P}\omega$ is the total function $\rho_{\mathrm{En}} : \omega^\omega \to \mathcal{P}\omega$ defined by

$$\rho_{\mathrm{En}}(x) = \{n \mid \exists k\ x_k = n + 1\}.$$

It is easy to see that ρ_{En} is an open admissible representation with Polish fibres.

As another example of an admissible representation of $\mathcal{P}\omega$ consider:

Example 5.22. Let $(s_n)_{n\in\omega}$ be an enumeration of the finite subsets of ω. We define $\rho_{<\infty} :\subseteq \omega^\omega \to \mathcal{P}\omega$ by

$$\rho_{<\infty}(\alpha) = x \quad \longleftrightarrow \quad \forall n \in \omega\ s_{\alpha(n)} \subseteq s_{\alpha(n+1)} \quad \text{and} \quad \bigcup_{n\in\omega} s_{\alpha(n)} = x.$$

The domain of $\rho_{<\infty}$ is closed and $\rho_{<\infty}$ is clearly continuous. The map $\rho_{<\infty}$ is also an admissible representation of the space $\mathcal{P}\omega$ since it is continuous and $\rho_{\mathrm{En}} \leqslant^{\mathrm{rep}} \rho_{<\infty}$, as witnessed by the continuous $f : \omega^\omega \to \operatorname{dom} \rho_{<\infty}$ defined by

$$f(\alpha)(n) = k, \quad \text{where } s_k = \{m \mid \exists j \leqslant n\ \alpha(j) = m + 1\}.$$

5.3.2 Relative continuity

The importance of admissible representations stems from the fact that continuity of a function between second countable T_0 spaces can be accounted for “in the codes”.

Definition 5.23. Let $\mathcal{X}, \mathcal{Y}$ be second countable T_0 spaces. We say that a total function $f : \mathcal{X} \to \mathcal{Y}$ is *relatively continuous* if for some (any) admissible representations $\rho_{\mathcal{X}}$ and $\rho_{\mathcal{Y}}$ of $\mathcal{X}$ and $\mathcal{Y}$ respectively, there exists a continuous $g : \operatorname{dom} \rho_{\mathcal{X}} \to \operatorname{dom} \rho_{\mathcal{Y}}$, called a *continuous realiser of* f, such that $f \circ \rho_{\mathcal{X}}(\alpha) = \rho_{\mathcal{Y}} \circ g(\alpha)$ for ever $\alpha \in \operatorname{dom} \rho_{\mathcal{X}}$.

Using the very definition of an admissible representation, it is easy to see that a function $f : \mathcal{X} \to \mathcal{Y}$ admits a continuous realiser for some choice of admissible representations of $\mathcal{X}$ and $\mathcal{Y}$ if and only if it admits a continuous realiser for any choice of admissible representations.

Theorem 5.24. *Let $\mathcal{X}, \mathcal{Y}$ be second countable T_0 spaces. A* total *function $f : \mathcal{X} \to \mathcal{Y}$ is relatively continuous if and only if f is continuous.*

Proof. Let $\rho_{\mathcal{X}}$ and $\rho_{\mathcal{Y}}$ be open admissible representations of $\mathcal{X}$ and $\mathcal{Y}$ respectively.

If $f : \mathcal{X} \to \mathcal{Y}$ is continuous, then $f \circ \rho_{\mathcal{X}} : \operatorname{dom} \rho_{\mathcal{X}} \to \mathcal{Y}$ is continuous. Since $\rho_{\mathcal{Y}}$ is admissible, there exists a partial continuous $g : \operatorname{dom} \rho_{\mathcal{X}} \to \operatorname{dom} \rho_{\mathcal{Y}}$ ($\operatorname{dom} f \circ \rho_{\mathcal{X}} = \operatorname{dom} \rho_{\mathcal{X}}$) with $f \circ \rho_{\mathcal{X}} = \rho_{\mathcal{Y}} \circ g$ on the domain of $\rho_{\mathcal{X}}$, so f is relatively continuous.

Conversely, if $f : \mathcal{X} \to \mathcal{Y}$ is relatively continuous there exists a partial continuous $g : \operatorname{dom} \rho_{\mathcal{X}} \to \operatorname{dom} \rho_{\mathcal{Y}}$ with $f \circ \rho_{\mathcal{X}} = \rho_{\mathcal{Y}} \circ g$ on $\operatorname{dom} f \circ \rho_{\mathcal{X}} = \operatorname{dom} \rho_{\mathcal{X}}$. Therefore $f \circ \rho_{\mathcal{X}} : \operatorname{dom} \rho_{\mathcal{X}} \to \mathcal{Y}$ is continuous. So the proof will be finished once we have established the following lemma. $\square$

Lemma 5.25. *Let $g :\subseteq \mathcal{X} \to \mathcal{Y}$ be a continuous, surjective and open map, and $f : \mathcal{Y} \to \mathcal{Z}$ any function. If $f \circ g :\subseteq \mathcal{X} \to \mathcal{Z}$ is continuous, then f is continuous.*

Proof. Let U be open in $\mathcal{Z}$. Then

$$\begin{aligned} f^{-1}(U) =& \{g(x) \mid x \in \operatorname{dom} g \wedge g(p) \in f^{-1}(U)\} && \text{since } g \text{ is onto,} \\ =& g((f \circ g)^{-1}(U)) \end{aligned}$$

is open in $\mathcal{Y}$ since g is an open map and $f \circ g$ is continuous. $\square$

5.3.3 Admissible representations and dimension

For an admissible representation $\rho :\subseteq \omega^\omega \to \mathcal{X}$ and a point $x \in \mathcal{X}$, one can think of $\alpha \in \omega^\omega$ with $\rho(\alpha) = x$ as a 'code' or 'name' for x. It is natural to ask what are the spaces which possess an injective admissible representation. It is actually simple to see that these spaces are exactly those of dimension zero. We now show this fact.

Recall the following fact on the cardinality of a base.

Lemma 5.26. *Let $\mathcal{X}$ be second countable. For every base $\mathcal{C}$, there is a countable base $\mathcal{C}' \subseteq \mathcal{C}$.*

Proof. Let (V_n) be countable base for $\mathcal{X}$. Whenever possible choose $C_{n,m} \in \mathcal{C}$ with $V_n \subseteq C_{n,m} \subseteq V_m$. Then the countable family of the $C_{n,m}$'s is a base for $\mathcal{X}$. Indeed for every $x \in V_m$ there is a $C \in \mathcal{C}$ with $x \in C \subseteq V_m$ (since $\mathcal{C}$ is a base for $\mathcal{X}$), and furthermore there exists n with $x \in V_n \subseteq C \subseteq V_m$ (since $(V_n)_{n\in\omega}$ is a base for $\mathcal{X}$), hence $x \in C_{n,m} \subseteq V_m$. □

Lemma 5.27. *Let $\mathcal{X}$ be second countable T_0 space and $\sigma :\subseteq \omega^\omega \to \mathcal{X}$ be an admissible representation of $\mathcal{X}$. Then there is $A \subseteq \operatorname{dom}\sigma$ such that $\sigma\restriction_A$ is an open admissible representation of $\mathcal{X}$.*

Proof. Let $\rho :\subseteq \omega^\omega \to \mathcal{X}$ be an open admissible representation of $\mathcal{X}$ which exists by Theorem 5.17. There exists a continuous $h : \operatorname{dom}\rho \to \operatorname{dom}\sigma$ that witnesses $\rho \leqslant^{\mathrm{rep}} \sigma$. We claim that $A = \{h(\alpha) \mid \alpha \in \operatorname{dom}\rho\}$ works. Indeed $\rho \leqslant^{\mathrm{rep}} \sigma\restriction_A$ as h also witnesses, and for every open $O \subseteq \omega^\omega$ we have

$$\sigma\restriction_A(O) = \{\sigma \circ h(\alpha) \mid \alpha \in \operatorname{dom}\rho\} = \rho(O). \qquad \square$$

Theorem 5.28. *Let $\mathcal{X}$ be a second countable T_0 space. The following are equivalent:*

(i) *$\mathcal{X}$ is 0-dimensional,*

(ii) *there exists an injective admissible representation of $\mathcal{X}$.*

Proof. (i)→(ii)*:* By Lemma 5.26, X admits a countable basis (V_n) consisting in clopen subsets of X, and for simplicity we may assume further that the basis is closed under complements, i.e. for every n there exists m with $X \setminus V_n = V_m$.

Let $\sigma :\subseteq \omega^\omega \to X$ be the partial map defined by $\sigma(\alpha) = x$ if and only if $\alpha : \omega \to 2$ is the characteristic function of $\{n \in \omega \mid x \in V_n\}$. Clearly σ is injective and continuous. To see that σ is admissible, it is enough to show that $\rho \leqslant^{\mathrm{rep}} \sigma$ where ρ is the standard representation of X with respect to (V_n). This is witnessed by the continuous function $g : \operatorname{dom}\rho \to \operatorname{dom}\sigma$ defined by

$$g(\alpha)(n) = \begin{cases} 1 & \text{if there exists } k \text{ with } \alpha(k) = n+1, \\ 0 & \text{if there exists } k \text{ with } \alpha(k) = m+1 \\ & \text{and } V_m = X \setminus V_n. \end{cases}$$

(ii)→(i)*:* Suppose that ρ is an injective admissible representation of $\mathcal{X}$. It follows from Lemma 5.27 that there exists $A \subseteq \operatorname{dom}\rho$ such that $\rho\restriction_A$ is open and admissible. But since an admissible representation is surjective and ρ is injective, we must have $A = \operatorname{dom}\rho$. Therefore ρ is a homeomorphism, and so $\mathcal{X}$ is homeomorphic to $\operatorname{dom}\rho$, hence $\mathcal{X}$ is 0-dimensional, as desired. $\square$

5.3.4 Relatively continuous relations

We have seen that a function $f : \mathcal{X} \to \mathcal{Y}$ between second countable T_0 spaces is continuous if and only if it is induced by some continuous function 'in the codes'. Moreover we have seen that when $\mathcal{X}$ is not 0-dimensional, then no admissible representation of $\mathcal{X}$ is injective, and so necessarily some points are to receive several codes. Since different codes of the same point can be sent onto codes of different points, a continuous function in the codes may very well induce a relation which is not functional on the spaces. Even though the resulting 'transformations' of the space are not necessarily functions, they are still continuous in some sense. They are called *relatively continuous relations*, and were first studied in [BH94].

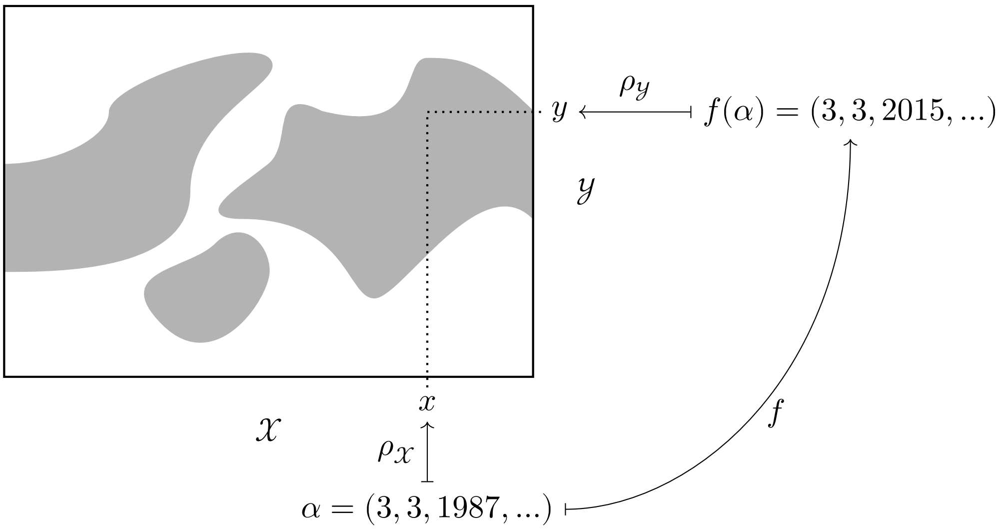


Figure 5.4: Relatively continuous relation.

Definition 5.29. Let $\mathcal{X}$ and $\mathcal{Y}$ be second countable T_0 spaces. A total relation $R : \mathcal{X} \rightrightarrows \mathcal{Y}$ is said to be *relatively continuous* if, for some (any) admissible representations $\rho_{\mathcal{X}}$ and $\rho_{\mathcal{Y}}$ of $\mathcal{X}$ and $\mathcal{Y}$ respectively, there exists a *continuous realiser* $f : \operatorname{dom}\rho_{\mathcal{X}} \to \operatorname{dom}\rho_{\mathcal{Y}}$ such that for every $\alpha \in \operatorname{dom}\rho_{\mathcal{X}}$ we have

$$(\rho_{\mathcal{X}}(\alpha), \rho_{\mathcal{Y}} \circ f(\alpha)) \in R.$$

Remark 5.30. Suppose $R : \mathcal{X} \rightrightarrows \mathcal{Y}$ is relatively continuous with respect to $\rho_{\mathcal{X}}$ and $\rho_{\mathcal{Y}}$ as witnessed by some continuous $f : \operatorname{dom}\rho_{\mathcal{X}} \to \rho_{\mathcal{Y}}$ and let $\sigma_{\mathcal{X}}, \sigma_{\mathcal{Y}}$ be admissible representations of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Since $\sigma_{\mathcal{X}} \leqslant^{\text{rep}} \rho_{\mathcal{X}}$ and $\rho_{\mathcal{Y}} \leqslant^{\text{rep}} \sigma_{\mathcal{Y}}$ there are continuous $g : \operatorname{dom}\sigma_{\mathcal{X}} \to \operatorname{dom}\rho_{\mathcal{X}}$ and $h : \operatorname{dom}\rho_{\mathcal{Y}} \to \operatorname{dom}\sigma_{\mathcal{Y}}$ with $\rho_{\mathcal{X}} \circ g = \sigma_{\mathcal{X}}$ and $\sigma_{\mathcal{Y}} \circ h = \rho_{\mathcal{Y}}$. Therefore if we set $f' : \operatorname{dom}\sigma_{\mathcal{X}} \to \operatorname{dom}\sigma_{\mathcal{Y}}$ to be $f' = h \circ f \circ g$ we obtain that for every $\alpha \in \operatorname{dom}\sigma_{\mathcal{X}}$

$$\sigma_{\mathcal{Y}} \circ f'(\alpha) = \sigma_{\mathcal{Y}} \circ h \circ f \circ g(\alpha) = \rho_{\mathcal{Y}} \circ f \circ g(\alpha).$$

Now since $\sigma_{\mathcal{X}}(\alpha) = \rho_{\mathcal{X}} \circ g(\alpha)$ if we let $\beta = g(\alpha)$ we have

$$(\sigma_{\mathcal{X}}(\alpha), \sigma_{\mathcal{Y}} \circ f'(\alpha)) = (\rho_{\mathcal{X}}(\beta), \rho_{\mathcal{Y}} \circ f(\beta)) \in R,$$

so R is relatively continuous with respect to $\sigma_{\mathcal{X}}$ and $\sigma_{\mathcal{Y}}$.

Clearly a function $f : \mathcal{X} \to \mathcal{Y}$ is (relatively) continuous if and only if its graph is relatively continuous as total relation from $\mathcal{X}$ to $\mathcal{Y}$. Moreover it is easily seen that the class of relatively continuous total relations is closed under composition.

Notice also that if $R : \mathcal{X} \rightrightarrows \mathcal{Y}$ is relatively continuous and $S : \mathcal{X} \rightrightarrows \mathcal{Y}$ is such that $R \subseteq S$, then S is relatively continuous too.

Let $\mathcal{X}$ and $\mathcal{Y}$ be second countable T_0 spaces together with admissible representations $\rho_{\mathcal{X}}$ and $\rho_{\mathcal{Y}}$. Every continuous function $f : \operatorname{dom}\rho_{\mathcal{X}} \to \operatorname{dom}\rho_{\mathcal{Y}}$ induces a total relation $R_f^{\rho_{\mathcal{X}},\rho_{\mathcal{Y}}} : \mathcal{X} \rightrightarrows \mathcal{Y}$ defined by

$$x \; R_f^{\rho_{\mathcal{X}},\rho_{\mathcal{Y}}} \; y \quad \longleftrightarrow \quad \exists \alpha \in \operatorname{dom}\rho_{\mathcal{X}} \, (\rho_{\mathcal{X}}(\alpha) = x \wedge \rho_{\mathcal{Y}} \circ f(\alpha) = y).$$

The function f witnesses that $R_f^{\rho_{\mathcal{X}},\rho_{\mathcal{Y}}}$ is relatively continuous. In fact, f witnesses that some $R : \mathcal{X} \rightrightarrows \mathcal{Y}$ is relatively continuous if and only if $R_f^{\rho_{\mathcal{X}},\rho_{\mathcal{Y}}} \subseteq R$. Therefore we have the following.

Fact 5.31. *Let $\mathcal{X}$ and $\mathcal{Y}$ be second countable T_0 and $\rho_{\mathcal{X}}$ and $\rho_{\mathcal{Y}}$ be admissible representations of $\mathcal{X}$ and $\mathcal{Y}$ respectively. A total relation $R : \mathcal{X} \rightrightarrows \mathcal{Y}$ is relatively continuous if and only if there exists a continuous $f : \operatorname{dom}\rho_{\mathcal{X}} \to \operatorname{dom}\rho_{\mathcal{Y}}$ such that $R_f^{\rho_{\mathcal{X}},\rho_{\mathcal{Y}}} \subseteq R$.*

From Theorem 5.28 and the previous fact, it follows that the relatively continuous relations from a 0-dimensional spaces are simply the continuously uniformisable relations.

Corollary 5.32. *Let $\mathcal{X}$ and $\mathcal{Y}$ be second countable T_0 with $\mathcal{X}$ 0-dimensional. A total relation R from $\mathcal{X}$ to $\mathcal{Y}$ is relatively continuous if and only if it admits a continuous uniformising function, i.e. there exists a continuous $f : \mathcal{X} \to \mathcal{Y}$ with $R(x, f(x))$ for all $x \in \mathcal{X}$.*

It is an interesting problem to look for an intrinsic characterisation of the relatively continuous total relations, that is, one which does not rely on the notion of admissible representation. Partial answers were obtained by Brattka and Hertling [BH94] and Pauly and Ziegler [PZ13]. However, to our knowledge, the general problem is still open. We conclude this subsection with some known results in the direction.

Let us say that $R : \mathcal{X} \rightrightarrows \mathcal{Y}$ *preserves open sets* if the set

$$R^{-1}(O) = \{x \in \mathcal{X} \mid \exists y \in O \; R(x, y)\}$$

is open in $\mathcal{X}$ for ever open set O of $\mathcal{Y}$.

Proposition 5.33 ([BH94, Proposition 4.5])**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be second countable T_0 spaces. There exists a class $\mathcal{R}$ of total relations from $\mathcal{X}$ to $\mathcal{Y}$ which preserves open sets such that for every $S : \mathcal{X} \rightrightarrows \mathcal{Y}$*

$$S \text{ is relatively continuous} \quad \longleftrightarrow \quad \exists R \in \mathcal{R} \quad R \subseteq S.$$

Proof. Let $\rho_{\mathcal{X}}$ and $\rho_{\mathcal{Y}}$ be admissible representations of $\mathcal{X}$ and $\mathcal{Y}$ respectively. By Theorem 5.17 we can choose $\rho_{\mathcal{X}}$ to be an open map. Let $\mathcal{R}$ be the family of total relations $R_f^{\rho_{\mathcal{X}},\rho_{\mathcal{Y}}}$ where $f : \operatorname{dom} \rho_{\mathcal{X}} \to \operatorname{dom} \rho_{\mathcal{Y}}$ is continuous. By Fact 5.31, it only remains to prove that $R_f^{\rho_{\mathcal{X}},\rho_{\mathcal{Y}}}$ preserves open sets for every continuous f.

Indeed for every continuous $f : \operatorname{dom} \rho_{\mathcal{X}} \to \operatorname{dom} \rho_{\mathcal{Y}}$ and every open O of $\mathcal{Y}$

$$(R_f^{\rho_{\mathcal{X}},\rho_{\mathcal{Y}}})^{-1}(O) = \{\rho_{\mathcal{X}}(x) \mid x \in (\rho_{\mathcal{Y}} \circ f)^{-1}(O)\} = \rho_{\mathcal{X}}[(\rho_{\mathcal{Y}} \circ f)^{-1}(O)],$$

which is open since $\rho_{\mathcal{Y}} \circ f$ is continuous and $\rho_{\mathcal{X}}$ is open. □

Moreover, in the case of a Polish codomain, Brattka and Hertling [BH94] showed the following.

Theorem 5.34. *Let $\mathcal{X}$ be second countable T_0, $\mathcal{Y}$ be Polish, and $R : \mathcal{X} \rightrightarrows \mathcal{Y}$ be such that $R^{\rightarrow}(x)$ is closed for every $x \in \mathcal{X}$. Then R is relatively continuous if and only if there exists $S : \mathcal{X} \rightrightarrows \mathcal{Y}$ that preserves open sets and such that $S \subseteq R$.*

Remark 5.35. One should notice that in general preserving open sets is not a sufficient condition for the relative continuity of a total relation. Consider for example the partition of ω^ω into

$$F = \{\alpha \in \omega^\omega \mid \exists n \forall k \geqslant n \ \alpha(k) = 0\} \quad \text{and} \quad F = \omega^\omega \setminus G.$$

Clearly G and F are both dense in ω^ω. Moreover it is well known that $F \in \mathbf{\Sigma}^0_2 \setminus \mathbf{\Pi}^0_2$. Consider the total relation $R = (G \times F) \cup (F \times G)$. Then $R^{-1}(O) = \omega^\omega$ for every non-empty open set, but R not relatively continuous. Indeed any function $f : \omega^\omega \to \omega^\omega$ which uniformises R needs to verify $f^{-1}(G) = F$, and since F is not $\mathbf{\Pi}^0_2$, f cannot be continuous.

5.3.5 Reduction by relatively continuous relations

Aiming to generalise the continuous reducibility outside the realm of the dimension zero while maintaining its nice properties, we first observed in Section 5.2 that the notion of reduction nicely generalises from functions to total relations. We have then singled out in Subsection 5.3.4 a class of total relations between second countable T_0 spaces which are continuous in a sense. We can now come to the definition of the notion of reducibility that we propose.

Definition 5.36. If $\mathcal{X}$ and $\mathcal{Y}$ are second countable T_0 spaces, $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$, we say that A is *relatively Wadge reducible* to B, in symbols $(A, \mathcal{X}) \preccurlyeq_{\mathrm{W}} (B, \mathcal{Y})$, if there exists a total relatively continuous $R : \mathcal{X} \rightrightarrows \mathcal{Y}$ that is a reduction of A to B.

We might sometimes make an abuse of notation and simply write $A \preccurlyeq_{\mathrm{W}} B$ in place of $(A, \mathcal{X}) \preccurlyeq_{\mathrm{W}} (B, \mathcal{Y})$ when the underlying spaces are understood.

Since the class of relatively continuous relations between second countable T_0 spaces contains the identity functions (in fact all continuous functions) and is closed under composition, $\preccurlyeq_{\mathrm{W}}$ is a quasi-order on the class of subsets of second countable T_0 spaces.

For every second countable T_0 spaces $\mathcal{X}$ and $\mathcal{Y}$, $(A, \mathcal{X}) \leqslant_W (B, \mathcal{Y})$ implies $(A, \mathcal{X}) \preccurlyeq_{\mathrm{W}} (B, \mathcal{Y})$, by Theorem 5.24. However, the qo $\preccurlyeq_{\mathrm{W}}$ should not be confused with the quasi-order $\leqslant_W$ of continuous reducibility.

The quasi-order $\preccurlyeq_{\mathrm{W}}$ intuitively relates to topological complexity in the following sense. Assume $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$ are subsets of second countable T_0 spaces. Consider the problem of deciding when given a point $x \in \mathcal{X}$ whether x belongs to A or not. Any admissible representation $\rho_{\mathcal{X}}$ of $\mathcal{X}$ provides an optimal way to represent this abstract problem in more concrete way, i.e. as a problem about infinite sequences of natural numbers, and therefore about infinite binary sequences. The membership problem in the codes for A is to decide when given a code $\alpha \in \operatorname{dom} \rho_{\mathcal{X}}$ for some point of $\mathcal{X}$ whether α is a code of a point in A or not. By the very definition of an admissible representation, the topological complexity of the membership problem in the codes for A does not depend on the chosen admissible representation. Therefore if $\rho_{\mathcal{X}}$ and $\rho_{\mathcal{Y}}$ are admissible representations of $\mathcal{X}$ and $\mathcal{Y}$ respectively, then $(A, \mathcal{X}) \preccurlyeq_{\mathrm{W}} (B, \mathcal{Y})$ if and only if one can continuously reduce the membership problem in the codes for A to the membership problem in the codes for B.

We state this observation in the following lemma which follows from Fact 5.31 and Fact 5.13.

Lemma 5.37. *Let $\mathcal{X}$ and $\mathcal{Y}$ be second countable T_0 spaces, with admissible representations $\rho_{\mathcal{X}}$ and $\rho_{\mathcal{Y}}$ respectively. For every $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$ the following are equivalent*

(i) $(A, \mathcal{X}) \preccurlyeq_W (B, \mathcal{Y})$,

(ii) $(\rho_{\mathcal{X}}^{-1}(A), \operatorname{dom} \rho_{\mathcal{X}}) \leqslant_W (\rho_{\mathcal{Y}}^{-1}(B), \operatorname{dom} \rho_{\mathcal{Y}})$.

In particular since 0-dimensional second countable spaces admits injective admissible representations, it follows that $\preccurlyeq_{\mathrm{W}}$ is a generalisation of $\leqslant_{\mathrm{W}}$ to spaces of arbitrary dimension.

Proposition 5.38. *Let $\mathcal{X}$ be a 0-dimensional separable metrisable space. Then $\leqslant_W$ and $\preccurlyeq_W$ coincide on subsets of $\mathcal{X}$.*

Proof. This follows from Corollary 5.32 and the previous lemma. □

It is an easy consequence of the results in Section 5.1 that the quasi-order $\preccurlyeq_{\mathrm{W}}$ is BQO and satisfies the Wadge Lemma when suitably restricted. We however postpone this discussion a little to consider a more pressing matter. Even though we have seen that the quasi-order $\preccurlyeq_{\mathrm{W}}$ conveys a certain a priori idea of topological complexity, it still remains to show that it refines the Borel and Hausdorff–Kuratowski hierarchies.

5.4 Borel and Hausdorff–Kuratowski hierarchies

The classical approach initiated by the French analysts Baire, Borel, and Lebesgue to the classification of the subsets of a metric space is more descriptive in nature. Sets are classified according to the complexity of their definition from open sets. This approach was continued later by Luzin, Suslin, Hausdorff, Sierpiński and Kuratowski.

As observed – apparently for the first time – by Tang [Tan79; Tan81], the classical definition of the Borel hierarchy in metric spaces is not satisfactory for non metrisable spaces. Following Selivanov [Sel06] and de Brecht [deB13] we use the following slightly modified definition of the Borel hierarchy in an arbitrary topological space (see also the paper by Spurný [Spu10]).

Definition 5.39. Let $\mathcal{X}$ be a topological space. For each positive ordinal $\alpha < \omega_1$ we define by induction

$$\begin{aligned}
\boldsymbol{\Sigma}^0_1(\mathcal{X}) &= \{O \subseteq \mathcal{X} \mid \ \mathcal{X} \text{ is open}\},\\
\boldsymbol{\Sigma}^0_\alpha(\mathcal{X}) &= \left\{ \bigcup_{i\in\omega} B_i \cap C_i^{\complement} \;\middle|\; B_i, C_i \in \bigcup_{\beta<\alpha} \boldsymbol{\Sigma}^0_\beta \text{ for each } i \in \omega \right\},\\
\boldsymbol{\Pi}^0_\alpha(\mathcal{X}) &= \{A^{\complement} \mid A \in \boldsymbol{\Sigma}^0_\alpha\},\\
\boldsymbol{\Delta}^0_\alpha(\mathcal{X}) &= \boldsymbol{\Sigma}^0_\alpha(\mathcal{X}) \cap \boldsymbol{\Pi}^0_\alpha(\mathcal{X}).
\end{aligned}$$

Proposition 5.40. *For any topological space $\mathcal{X}$ and any $\alpha > 0$:*

(i) *$\boldsymbol{\Sigma}^0_\alpha(\mathcal{X})$ is closed under countable union and finite intersection;*

(ii) *$\boldsymbol{\Pi}^0_\alpha(\mathcal{X})$ is closed under countable intersection and finite union;*

(iii) *$\boldsymbol{\Delta}^0_\alpha(\mathcal{X})$ is closed under finite union and intersection as well as under complementation.*

Proposition 5.41. *If $\alpha < \beta$, then $\boldsymbol{\Sigma}^0_\alpha \cup \boldsymbol{\Pi}^0_\alpha \subseteq \boldsymbol{\Delta}^0_\beta$. So the following diagram of inclusion holds between Borel classes:*

$$\begin{tikzcd}
 & \boldsymbol{\Sigma}^0_1 \arrow[dr, phantom, sloped, "\supsetneq"] & & \boldsymbol{\Sigma}^0_2 \arrow[dr, phantom, sloped, "\supsetneq"] & & & & \boldsymbol{\Sigma}^0_\alpha \arrow[dr, phantom, sloped, "\supsetneq"] & & \\
\boldsymbol{\Delta}^0_1 \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \boldsymbol{\Delta}^0_2 \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \boldsymbol{\Delta}^0_3 \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & \cdots & \boldsymbol{\Delta}^0_\alpha \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \boldsymbol{\Delta}^0_{\alpha+1} \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & \cdots \\
 & \boldsymbol{\Pi}^0_1 \arrow[ur, phantom, sloped, "\subsetneq"] & & \boldsymbol{\Pi}^0_2 \arrow[ur, phantom, sloped, "\subsetneq"] & & & & \boldsymbol{\Pi}^0_\alpha \arrow[ur, phantom, sloped, "\subsetneq"] & &
\end{tikzcd}$$

Figure 5.5: The Borel hierarchy

Proposition 5.42. *If $\alpha > 2$, then*

$$\boldsymbol{\Sigma}^0_\alpha(\mathcal{X}) = \left\{ \bigcup_{i\in\omega} B_i \;\middle|\; B_i \in \bigcup_{\beta<\alpha} \boldsymbol{\Pi}^0_\beta(\mathcal{X}) \text{ for each } i \in \omega \right\}.$$

And if $\mathcal{X}$ is metrisable the previous statement holds also for $\alpha = 2$, i.e.

$$\mathbf{\Sigma}^0_2(\mathcal{X}) = \left\{ \bigcup_{i\in\omega} B_i \;\middle|\; B_i \in \mathbf{\Pi}^0_1(\mathcal{X}) \textit{ for each } i \in \omega \right\}.$$

Hausdorff and later Kuratowski refined the Borel hierarchy by introducing the so called *difference Hierarchy*. Recall that any ordinal α can uniquely be expressed as $\alpha = \lambda + n$ where λ is limit or equal to 0, and $n < \omega$. The ordinal α is said to be *even* if n is even, otherwise α is said to be *odd*.

Definition 5.43. Let $\xi \geqslant 1$ be a countable ordinal. For any sequence $(C_\eta)_{\eta<\xi}$ with $\alpha < \beta < \xi$ implies $C_\alpha \subseteq C_\beta$, the set $A = D_\xi((C_\eta)_{\eta<\xi})$ is defined by

$$A = \begin{cases} \bigcup\{C_\eta \setminus \bigcup_{\eta'<\eta} C_{\eta'} \mid \eta \text{ odd}, \eta < \xi\} & \text{for } \xi \text{ even,} \\ \bigcup\{C_\eta \setminus \bigcup_{\eta'<\eta} C_{\eta'} \mid \eta \text{ even}, \eta < \xi\} & \text{for } \xi \text{ odd.} \end{cases}$$

For a topological space $\mathcal{X}$, $1 \leqslant \alpha < \omega_1$ and $1 \leqslant \xi < \omega_1$ we let $D_\xi(\mathbf{\Sigma}^0_\alpha(\mathcal{X}))$ be the class of all sets $D_\xi((C_\eta)_{\eta<\xi})$ where $(C_\eta)_{\eta<\xi}$ is an increasing sequence in $\mathbf{\Sigma}^0_\alpha(\mathcal{X})$. Notice that in particular $D_1(\mathbf{\Sigma}^0_\alpha) = \mathbf{\Sigma}^0_\alpha$.

Of course if $f : \mathcal{X} \to \mathcal{Y}$ is a continuous map and $A \in D_\xi(\mathbf{\Sigma}^0_\alpha(\mathcal{Y}))$, then $f^{-1}(A) \in D_\xi(\mathbf{\Sigma}^0_\alpha(\mathcal{X}))$. For the Baire space, this straightforward observation is crystallised in the definition of a *pointclass*, that is a collection of subsets of the Baire space closed under continuous preimages, or in other words, an initial segment of the Wadge quasi-order on the Baire space. In any topological space, the fact that the classes $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Sigma}^0_\alpha$ and $D_\xi(\mathbf{\Sigma}^0_\alpha)$ are closed under continuous preimages means that these classes are initial segment of the quasi-order of continuous reducibility $\leqslant_{\mathrm{W}}$. In this sense $\leqslant_{\mathrm{W}}$ refines these classical hierarchies.

We now show that in an arbitrary second countable T_0 space $\mathcal{X}$ the classes $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Sigma}^0_\alpha$, $D_\xi(\mathbf{\Sigma}^0_\alpha)$ enjoy the stronger and less straightforward property of being initial segments of the quasi-order $\preccurlyeq_{\mathrm{W}}$. Therefore the quasi-order $\preccurlyeq_{\mathrm{W}}$ also refines these classical hierarchies.

Proposition 5.44. *Let $\mathcal{X}$ and $\mathcal{Y}$ be second countable T_0 spaces and $A \subseteq \mathcal{X}$, $B \subseteq \mathcal{Y}$. For every $1 \leqslant \alpha, \xi < \omega_1$,*

(i) *if $B \in \mathbf{\Sigma}^0_\alpha(\mathcal{Y})$ and $A \preccurlyeq_W B$, then $A \in \mathbf{\Sigma}^0_\alpha(\mathcal{X})$,*

(ii) *if $B \in D_\xi(\mathbf{\Sigma}^0_\alpha(\mathcal{Y}))$ and $A \preccurlyeq_W B$, then $A \in D_\xi(\mathbf{\Sigma}^0_\alpha(\mathcal{X}))$.*

We defer the proof of the previous proposition until the end of this section since it follows from results of independent interest. The proof relies essentially on the following proposition which is a slightly modified version of a result due to Saint Raymond [Sai07, Lemma 17]. Its relevance in our context was first observed by de Brecht [deB13]. It is based on Baire category and we refer the reader to the textbook by Kechris [Kec95] for the basic definitions and results.

Proposition 5.45. *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces, with $\mathcal{X}$ metrisable. Let $\varphi : \mathcal{X} \to \mathcal{Y}$ be an open, continuous map with Polish fibres, i.e. $\varphi^{-1}(y)$ is Polish for all $y \in \mathcal{Y}$. For every $Z \subseteq \mathcal{X}$ define:*

$$N_0(Z) = \{y \in \mathcal{Y} \mid Z \cap \varphi^{-1}(y) \text{ is non meagre in } \varphi^{-1}(y)\},$$
$$N_1(Z) = \{y \in \mathcal{Y} \mid Z \cap \varphi^{-1}(y) \text{ is comeagre in } \varphi^{-1}(y)\}.$$

Then for every positive ordinal $\alpha < \omega_1$,

(i) *If $Z \in \boldsymbol{\Sigma}^0_\alpha(\mathcal{X})$, then $N_0(Z) \in \boldsymbol{\Sigma}^0_\alpha(\mathcal{Y})$,*

(ii) *If $Z \in \boldsymbol{\Pi}^0_\alpha(\mathcal{X})$, then $N_1(Z) \in \boldsymbol{\Pi}^0_\alpha(\mathcal{Y})$.*

In particular, if φ is further assumed to be surjective then for every $A \subseteq \mathcal{Y}$ and every positive ordinal $\alpha < \omega_1$,

(i) $\varphi^{-1}(A) \in \boldsymbol{\Sigma}^0_\alpha(\mathcal{X}) \quad \longleftrightarrow \quad A \in \boldsymbol{\Sigma}^0_\alpha(\mathcal{Y})$,

(ii) $\varphi^{-1}(A) \in \boldsymbol{\Pi}^0_\alpha(\mathcal{X}) \quad \longleftrightarrow \quad A \in \boldsymbol{\Pi}^0_\alpha(\mathcal{Y})$.

Proof. Since $N_1(\mathcal{X} \setminus Z) = \mathcal{Y} \setminus N_0(Z)$ both statements are equivalent for every α. Let $(V_k)_{k\in\omega}$ be a countable base for the topology of $\mathcal{X}$. We proceed by induction on α.

For $\alpha = 1$ let $Z \in \boldsymbol{\Sigma}^0_1$, since φ is assumed to be open we have $\varphi(Z)$ is open in $\mathcal{Y}$. Since $\varphi^{-1}(y)$ is a Baire space for all $y \in \mathcal{Y}$, the open subset $Z \cap \varphi^{-1}(y)$ of $\varphi^{-1}(y)$ is non meagre if and only if it is non empty. So $N_0(Z) = \varphi(Z) \in \boldsymbol{\Sigma}^0_1(\mathcal{Y})$.

So assume now that both statements are true for every $\alpha' < \alpha$ and let $Z \in \boldsymbol{\Sigma}^0_\alpha$. Since $\mathcal{X}$ is metrisable, Z is the union of a countable family $(Z_n)_{n\in\omega}$ with $Z_n \in \boldsymbol{\Pi}^0_{\alpha_n}$ for some $\alpha_n < \alpha$. For any point $y \in \mathcal{Y}$, using the fact that any Borel subset of a Polish space has the Baire Property, we have the following equivalences:

$$\begin{aligned} & Z \cap \varphi^{-1}(y) \text{ is non meagre in } \varphi^{-1}(y) \\ \leftrightarrow\ & \exists n\ Z_n \cap \varphi^{-1}(y) \text{ is non meagre in } \varphi^{-1}(y) \\ \leftrightarrow\ & \exists n\ Z_n \cap \varphi^{-1}(y) \text{ is comeagre in some non-empty open subset of } \varphi^{-1}(y) \\ \leftrightarrow\ & \exists n\, \exists k\ (Z_n \cup V_k^{\complement}) \cap \varphi^{-1}(y) \text{ is comeagre in } \varphi^{-1}(y) \text{ and } V_k \cap \varphi^{-1}(y) \neq \emptyset. \end{aligned}$$

Therefore,

$$N_0(Z) = \bigcup_{n,k} N_1(Z_n \cup V_k^{\complement}) \cap \varphi(V_k).$$

Now $Z_n \cup V_k^{\complement} \in \mathbf{\Pi}^0_{\alpha_n}$, and so $N_1(Z_n \cup V_k^{\complement}) \in \mathbf{\Pi}^0_{\alpha_n}$ by the induction hypothesis. Moreover $\varphi(V_k) \in \mathbf{\Sigma}^0_1$ since φ is an open map. It follows that $N_0(Z)$ is $\mathbf{\Sigma}^0_\alpha$ according to Definition 5.39.

For the second claim, assume that φ is moreover surjective and notice that if $A \subseteq \mathcal{Y}$ then for $Z = \varphi^{-1}(A)$ we have $A = N_0(Z) = N_1(Z)$. □

Building on Proposition 5.45 and using the same technique, de Brecht [deB13] showed:

Proposition 5.46. *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces, with $\mathcal{X}$ metrisable. Let $\varphi : \mathcal{X} \to \mathcal{Y}$ be an open and continuous map with Polish fibres, $A \subseteq \mathcal{Y}$ and $1 \leqslant \alpha, \xi < \omega_1$. If $\varphi^{-1}(A) = D_\xi((C_\eta)_{\eta<\xi})$ with $(C_\eta)_{\eta<\xi}$ an increasing sequence in $\mathbf{\Sigma}^0_\alpha$, then $A = D_\xi(N_0(C_\eta)_{\eta<\xi})$. So $A \in D_\xi(\mathbf{\Sigma}^0_\alpha(\mathcal{Y}))$ if and only if $\varphi^{-1}(A) \in D_\xi(\mathbf{\Sigma}^0_\alpha(\mathcal{X}))$.*

Proof. Let $B_\eta = N_0(C_\eta)$. First let $y \in A$. Since $\varphi^{-1}(y) = \bigcup_{\eta<\xi} C_\eta \cap \varphi^{-1}(y)$ and $\varphi^{-1}(y)$ is Polish and non empty, there exists a least $\eta_y < \xi$ such that $C_{\eta_y} \cap \varphi^{-1}(y)$ is non meagre in $\varphi^{-1}(y)$, i.e. $y \in B_{\eta_y}$. In particular, $C_{\eta'} \cap \varphi^{-1}(y)$ is meagre in $\varphi^{-1}(y)$ for all $\eta' < \eta_y$, hence $\bigcup_{\eta'<\eta_y} C_{\eta'} \cap \varphi^{-1}(y)$ is meagre in $\varphi^{-1}(y)$. It follows that $(C_{\eta_y} \setminus \bigcup_{\eta'<\eta_y} C_{\eta'}) \cap \varphi^{-1}(y)$ is non meagre in $\varphi^{-1}(y)$, so in particular it contains some $x \in \mathcal{X}$. Since $x \in \varphi^{-1}(A) = D_\xi((C_\eta)_{\eta<\xi})$ the parity of ξ must differ from that of η_y. Therefore $y \in D_\xi((B_\eta)_{\eta<\xi})$.

Conversely let $y \in D_\xi((B_\eta)_{\eta<\xi})$. There exists $\eta_y < \xi$ whose parity is different from that of ξ such that $y \in B_{\eta_y} \setminus \bigcup_{\eta'<\eta_y} B_{\eta'}$. Since $B_\eta = N_0(A_\eta)$, $C_{\eta_y} \cap \varphi^{-1}(y)$ is non meagre in $\varphi^{-1}(y)$, and $\bigcup_{\eta'<\eta_y} C_{\eta'} \cap \varphi^{-1}(y)$ is meagre in $\varphi^{-1}(y)$. As before $(C_{\eta_y} \setminus \bigcup_{\eta'<\eta_y} C_{\eta'}) \cap \varphi^{-1}(y)$ is non meagre in $\varphi^{-1}(y)$ and so in particular it must contain some point $x \in \mathcal{X}$. We have $x \in D_\xi((C_\eta)_{\eta<\xi}) = \varphi^{-1}(A)$ and so $y = \varphi(x) \in A$. □

Since every second countable T_0 space has an admissible representation which is open and has Polish fibres, we obtain:

Theorem 5.47 ([deB13, Theorem 78])**.** *Let $\mathcal{X}$ be a second countable T_0 space, $\rho :\subseteq \omega^\omega \to \mathcal{X}$ an admissible representation of $\mathcal{X}$. For any countable $\alpha, \xi > 0$ and every $A \subseteq \mathcal{X}$ we have*

$$A \in D_\xi(\mathbf{\Sigma}^0_\alpha(\mathcal{X})) \quad \longleftrightarrow \quad \rho^{-1}(A) \in D_\xi(\mathbf{\Sigma}^0_\alpha(\operatorname{dom} \rho)).$$

Proof. The left to right implication follows from the continuity of the admissible representation and the fact that the preimage map ρ^{-1} is a complete Boolean homomorphism.

For the right to left implication, it is enough by Propositions 5.45 and 5.46 to show that we can assume ρ to be open with Polish fibres – since such an admissible representation always exists by Theorem 5.17. So let $\delta :\subseteq \omega^\omega \to \mathcal{X}$ be any admissible representation of $\mathcal{X}$, then there exists a continuous $f : \operatorname{dom}\rho \to \operatorname{dom}\sigma$ with $\delta \circ f = \rho$ on the domain of ρ. If $\delta^{-1}(A) \in D_\xi(\mathbf{\Sigma}^0_\alpha(\operatorname{dom}\delta))$ then as in the first implication we have

$$\rho^{-1}(S) = f^{-1}(\delta^{-1}(S)) \in D_\alpha(\mathbf{\Sigma}^0_\theta(\operatorname{dom}\rho)).$$

This concludes the claim. $\square$

The proof of Proposition 5.44 is now straightforward.

Proof of Proposition 5.44. Since $D_1(\mathbf{\Sigma}^0_\alpha)$ is just $\mathbf{\Sigma}^0_\alpha$, (i) is a particular case of (ii). Let $B \in D_\xi(\mathbf{\Sigma}^0_\alpha(\mathcal{Y}))$ and suppose that $A \subseteq \mathcal{X}$ satisfies $A \preccurlyeq_{\mathrm{W}} B$. Let $\rho_\mathcal{X}, \rho_\mathcal{Y}$ be admissible representations of $\mathcal{X}, \mathcal{Y}$ respectively. Since $A \preccurlyeq_{\mathrm{W}} B$, there exists a continuous $f : \operatorname{dom}\rho_\mathcal{X} \to \operatorname{dom}\rho_\mathcal{Y}$ with $(\rho_\mathcal{Y} \circ f)^{-1}(B) = \rho_\mathcal{X}^{-1}(A)$. By continuity, $\rho_\mathcal{X}^{-1}(A) = (\rho_\mathcal{Y} \circ f)^{-1}(B) \in D_\xi(\mathbf{\Sigma}^0_\alpha(\operatorname{dom}\rho_\mathcal{X}))$, and so by Theorem 5.47 A is $D_\xi(\mathbf{\Sigma}^0_\alpha)$ in $\mathcal{X}$. $\square$

In the Baire space, the pointclasses and the Wadge quasi-order are two sides of the same coin. Moreover while the pointclasses $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ and $D_\xi(\mathbf{\Sigma}^0_\alpha)$ are defined in terms of operations on open sets, Wadge showed that every *non-self-dual* Borel pointclass – i.e. a pointclass of Borel sets which is not closed under complementation – can be described in this fashion. Since we investigate a generalisation of the Wadge quasi-order on ω^ω to more general spaces, it is therefore natural to wonder what is the generalisation of the notion pointclass. With Proposition 5.44 in mind and by analogy with the case of ω^ω, our approach suggests to define a 'pointclass' in any second countable T_0 space as an initial segment of the quasi-order $\preccurlyeq_{\mathrm{W}}$.

It is worth mentioning that after the paper on which this chapter is based was finished we discovered that Louveau and Saint Raymond [LS11, Section 4] define for each Borel non-self dual pointclass Γ in ω^ω a corresponding class $\Gamma(\mathcal{X})$ for every metric separable space $\mathcal{X}$ as follows, $\Gamma(\mathcal{X})$ is the family of those sets $A \subseteq \mathcal{X}$ such that for every (total) continuous map $f : \omega^\omega \to \mathcal{X}$ we have $f^{-1}(A) \in \Gamma$. It is easy to see these classes $\Gamma(\mathcal{X})$ are always initial segments of our quasi-order $\preccurlyeq_{\mathrm{W}}$. Conversely when $\mathcal{X}$ is Polish, one can show[1]

[1] Using that every Polish space has a total (i.e. defined on the whole ω^ω) admissible representation as proved by Brattka [Bra99, Corollary 4.4.12].

that if Γ' is a initial segment for $\preccurlyeq_{\mathrm{W}}$ consisting in Borel subsets of $\mathcal{X}$ which is not closed under complementation, then there is a Borel non-self-dual point class Γ of ω^ω such that $\Gamma' = \Gamma(\mathcal{X})$. However we reserve the investigation of the relation between these classes and the quasi-order $\preccurlyeq_{\mathrm{W}}$ as well as a general discussion on the notion of pointclass in arbitrary spaces for a later work.

5.5 A general reduction game

We now explain how the results of Section 5.1 imply that the quasi-order $\preccurlyeq_{\mathrm{W}}$ is BQO and satisfies the Wadge Lemma when suitably restricted.

In the investigation of the quasi-order $\preccurlyeq_{\mathrm{W}}$ the following simple adaptation of the game in Definition 5.1 is essential.

Definition 5.48. Let $\mathcal{X}$ and $\mathcal{Y}$ be second countable T_0 spaces, $\rho_{\mathcal{X}}$, $\rho_{\mathcal{Y}}$ admissible representations of $\mathcal{X}$ and $\mathcal{Y}$ respectively, and $A \subseteq \mathcal{X}$, $B \subseteq \mathcal{Y}$. We define a perfect information two players game $G^{\rho_{\mathcal{X}},\rho_{\mathcal{Y}}}(A, B)$ as follows. Player I starts by choosing some $\alpha_0 \in \omega$ and then Player II chooses some $\beta_0 \in \omega$, then Player I choose some $\alpha_1 \in \omega$, so on and so forth. Player II wins the play (α, β) if and only if

$$\alpha \in \operatorname{dom}\rho_{\mathcal{X}} \text{ implies } \Big[\beta \in \operatorname{dom}\rho_{\mathcal{Y}} \text{ and } \big(\rho_{\mathcal{X}}(\alpha) \in A \leftrightarrow \rho_{\mathcal{Y}}(\beta) \in B\big)\Big].$$

When $\rho_{\mathcal{X}} = \rho_{\mathcal{Y}}$ we write $G^{\rho_{\mathcal{X}}}(A, B)$ instead of $G^{\rho_{\mathcal{X}},\rho_{\mathcal{X}}}(A, B)$.

Of course the game $G^{\rho_{\mathcal{X}},\rho_{\mathcal{Y}}}(A, B)$ is tightly related to the reducibility by relatively continuous relations.

Proposition 5.49. *Let $\mathcal{X}$ and $\mathcal{Y}$ be second countable T_0 spaces, $\rho_{\mathcal{X}}, \rho_{\mathcal{Y}}$ be admissible representations of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then for all $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$:*

(i) *If Player* II *has a winning strategy in $G^{\rho_{\mathcal{X}},\rho_{\mathcal{Y}}}(A, B)$, then $A \preccurlyeq_W B$,*

(ii) *If Player* I *has a winning strategy in $G^{\rho_{\mathcal{X}},\rho_{\mathcal{Y}}}(A, B)$, then $B \preccurlyeq_W A^{\complement}$.*

Proof. A winning strategy for Player II induces a total continuous function $f : \omega^\omega \to \omega^\omega$ such that for every $\alpha \in \operatorname{dom}\rho_{\mathcal{X}}$, we have $f(\alpha) \in \operatorname{dom}\rho_{\mathcal{Y}}$ and

$$\rho_{\mathcal{X}}(\alpha) \in A \quad \longleftrightarrow \quad \rho_{\mathcal{Y}}(f(\alpha)) \in B,$$

or equivalently

$$\alpha \in \rho_{\mathcal{X}}^{-1}(A) \quad \longleftrightarrow \quad f(\alpha) \in \rho_{\mathcal{Y}}^{-1}(B).$$

Therefore if Player II has a winning strategy in $G^{\rho_{\mathcal{X}},\rho_{\mathcal{Y}}}(A,B)$, then

$$(\rho_{\mathcal{X}}^{-1}(A), \operatorname{dom}\rho_{\mathcal{X}}) \leqslant_{\mathrm{W}} (\rho_{\mathcal{Y}}^{-1}(B), \operatorname{dom}\rho_{\mathcal{Y}})$$

and so $A \preccurlyeq_{\mathrm{W}} B$ by Lemma 5.37.

Now any winning strategy for Player I induces a continuous function $g : \omega^\omega \to \omega^\omega$ such that whenever $\alpha \in \operatorname{dom}\rho_{\mathcal{Y}}$ then $g(\alpha) \in \operatorname{dom}\rho_{\mathcal{X}}$ and

$$\rho_{\mathcal{Y}}(\alpha) \in B \quad \longleftrightarrow \quad \rho_{\mathcal{X}}(g(\alpha)) \in A^{\complement}$$

or equivalently,

$$g(\alpha) \notin \rho_{\mathcal{X}}^{-1}(A) \quad \longleftrightarrow \quad \alpha \in \rho_{\mathcal{Y}}^{-1}(B)$$

Therefore if Player I has a winning strategy in $G^{\rho_{\mathcal{X}},\rho_{\mathcal{Y}}}(A,B)$, then $(B,\mathcal{Y}) \preccurlyeq_{\mathrm{W}} (A^{\complement},\mathcal{X})$. □

Remark 5.50. A priori the existence of a winning strategy for Player II in $G^{\rho_{\mathcal{X}},\rho_{\mathcal{Y}}}(A,B)$ is *not* equivalent to $A \preccurlyeq_{\mathrm{W}} B$. This is however the case when $\rho_{\mathcal{Y}}$ satisfies a stronger property than being an admissible representation.

Recall that a a partial map $h :\subseteq \omega^\omega \to \omega^\omega$ is called *Lipschitz* if for every $\alpha,\beta \in \operatorname{dom} h$ and every $n \in \omega$, $\alpha{\restriction}_n = \beta{\restriction}_n$ implies $h(\alpha){\restriction}_n = h(\beta){\restriction}_n$.

Let us say that a partial continuous map $\rho :\subseteq \omega^\omega \to \mathcal{Y}$ is a *nice admissible representation* if for every partial continuous function $f :\subseteq \omega^\omega \to \mathcal{Y}$ there exists a *Lipschitz map* $h : \operatorname{dom} f \to \operatorname{dom}\rho$ such that for every $\alpha \in \operatorname{dom} f$ we have $f(\alpha) = \rho(h(\alpha))$. Since Lipschitz maps are continuous, a nice admissible representation is in particular an admissible representation. Moreover, as we noticed in Remark 5.18 *ii.*, the standard representation of a second countable T_0 space $\mathcal{Y}$ relatively to some base is always a *nice admissible representation* of $\mathcal{Y}$. Hence every second countable T_0 space has a nice admissible representation.

Now if $\rho_{\mathcal{Y}}$ is a nice admissible representation of $\mathcal{Y}$, then Player II has a winning strategy in $G^{\rho_{\mathcal{X}},\rho_{\mathcal{Y}}}(A,B)$ if and only if $A \preccurlyeq_{\mathrm{W}} B$. Indeed if $A \preccurlyeq_{\mathrm{W}} B$, then there exists some relatively continuous reduction of A to B, and therefore there exists some continuous function $F' : \operatorname{dom}\rho_{\mathcal{X}} \to \operatorname{dom}\rho_{\mathcal{Y}}$. Since $\rho_{\mathcal{Y}}$ is a nice admissible representation there exists a Lipschitz map $F : \operatorname{dom}\rho_{\mathcal{X}} \to \operatorname{dom}_{\rho_{\mathcal{Y}}}$ such that $\rho_{\mathcal{Y}} \circ F' = \rho_{\mathcal{Y}} \circ F$. This means that $F(\alpha){\restriction}_n$ depends only on $\alpha{\restriction}_n$ for every $\alpha \in \operatorname{dom}\rho_{\mathcal{X}}$. Therefore a winning strategy for Player II in $G^{\rho_{\mathcal{X}},\rho_{\mathcal{Y}}}(A,B)$ is for example given by the function $\sigma : \omega^{<\omega} \to \omega^{<\omega}$ defined by induction on the length by:

$$\sigma(\emptyset) = \emptyset$$
$$\sigma(s) = \begin{cases} F(\alpha){\restriction}_{|s|} & \text{for some } \alpha \in N_s \cap \operatorname{dom} \rho_{\mathcal{X}}, \text{ if any,} \\ \sigma(s{\restriction}_k) ^\frown 0^{|s|-k} & \text{if } N_s \cap \operatorname{dom} \rho_{\mathcal{X}} = \emptyset \text{ and } k < |s| \text{ is the largest} \\ & \text{such that } N_{s{\restriction}_k} \cap \operatorname{dom} \rho_{\mathcal{X}} \neq \emptyset. \end{cases}$$

Observe that when $\operatorname{dom} \rho_{\mathcal{X}}$ and $\operatorname{dom} \rho_{\mathcal{Y}}$ are Borel subsets of ω^ω, the reduction game $G^{\rho_{\mathcal{X}}, \rho_{\mathcal{Y}}}(A, B)$ is Borel and therefore determined by Martin's result as long as $A \subseteq \mathcal{X}$, $B \subseteq \mathcal{Y}$ are Borel subsets of $\mathcal{X}$ and $\mathcal{Y}$ respectively. We are therefore naturally led to the following definition.

Definition 5.51. A second countable T_0 space $\mathcal{X}$ is called *Borel representable* if there exists an admissible representation $\rho :\subseteq \omega^\omega \to \mathcal{X}$ of $\mathcal{X}$ such that $\operatorname{dom} \rho$ Borel in ω^ω.

It is clear that every Borel subset of the Scott domain $\mathcal{P}\omega$ is Borel representable. We do not know if the converse is true, namely:

Problem 3. *Is every Borel representable space homeomorphic to a Borel subset of $\mathcal{P}\omega$?*

From Proposition 5.49 and Borel determinacy we obtain the following generalisation of Lemma 5.10.

Theorem 5.52 (Generalised Wadge Lemma)**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be a Borel representable spaces, $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$ be Borel. Then*

$$\textit{either } (A, \mathcal{X}) \preccurlyeq_W (B, \mathcal{Y}) \textit{ or } (B, \mathcal{Y}) \preccurlyeq_W (A^\complement, \mathcal{X}).$$

For the same reasons already exposed after Lemma 5.10, the quasi-order $\preccurlyeq_{\mathrm{W}}$ has a very simple structure when restricted to Borel subsets of Borel representable spaces. Namely antichains are of size at most two, and up to equivalence they are necessarily of the form $(A, \mathcal{X})$ and $(A^\complement, \mathcal{X})$. Moreover if we stipulate that $(A, \mathcal{X})$ and $(A^\complement, \mathcal{X})$ are always equivalent, this makes $\preccurlyeq_{\mathrm{W}}$ into a linear order.

Using the above game, the well-foundedness of $\preccurlyeq_{\mathrm{W}}$ on Borel subsets of Borel representable spaces can be obtained by following the proof of the well-foundedness of the Wadge quasi-order on ω^ω as for example presented by Kechris [Kec95, p. 21.15]. However we think it is best seen as a particular case of the following extension of Theorem 5.9 to Borel representable spaces.

Let us first extend Definition 5.8 in the obvious way:

Definition 5.53. Let Q be a quasi-order. A map $l : \mathcal{X} \to Q$ from some Borel representable space $\mathcal{X}$ is called a *Borel Q-labelling function of $\mathcal{X}$* if

(1) for every $q \in Q$, the set $l^{-1}(q)$ is Borel in $\mathcal{X}$,

(2) the image $\operatorname{Im} l$ of l is countable.

We refer to $\mathcal{X}$ as the domain of l and denote it by $\operatorname{dom} l$. We let $B^{\text{gen}}(Q)$ be the class of all Borel Q-labelling of some Borel representable space.

We quasi-order $B^{\text{gen}}(Q)$ by

$$
\begin{aligned}
l_0 \preccurlyeq_{\mathrm{c}} l_1 \quad \longleftrightarrow \quad & \text{there exists a relatively continuous } R : \operatorname{dom} l_0 \rightrightarrows \operatorname{dom} l_1 \\
& \text{such that for every } x \in \operatorname{dom} l_0 \text{ and every } y \in \operatorname{dom} l_1, \\
& x \mathrel{R} y \text{ implies } l_0(x) \leqslant l_1(g(x)) \text{ in } Q.
\end{aligned}
$$

We now have the following straightforward generalisation of Theorem 5.9.

Theorem 5.54. *If Q is* BQO, *then $B^{gen}(Q)$ is* BQO.

Proof. Let $h : [\omega]^\infty \to B^{\text{gen}}(Q)$ be a locally constant multi-sequence. We can choose for every $l \in B^{\text{gen}}(Q)$ in the countable image of h an admissible representation $\rho_{\operatorname{dom} l} : D_l \to \operatorname{dom}_l$ with D_l a Borel subset of ω^ω. We get a multi-sequence $\tilde{h} : [\omega]^\infty \to \mathfrak{L}^B(Q)$ by letting $\tilde{h}(X) = h(X) \circ \rho_{\operatorname{dom} h(X)}$. Since $\mathfrak{L}^B(Q)$ is BQO by Theorem 5.7, there exists $X \in [\omega]^\infty$ such that $\tilde{h}(X) \leqslant_{\mathfrak{L}} \tilde{h}({}_*X)$. To lighten the notation, let $h(X) = l_0 : \mathcal{X}_0 \to Q$ and $h({}_*X) = l_1 : \mathcal{X}_1 \to Q$ so that we have $l_0 \circ \rho_{\mathcal{X}_0} \leqslant_{\mathfrak{L}} l_1 \circ \rho_{\mathcal{X}_1}$, i.e. Player II has a winning strategy σ in $G(l_0 \circ \rho_{\mathcal{X}_0}, l_1 \circ \rho_{\mathcal{X}_1})$. Now this winning strategy for Player II induces a continuous function $\sigma^* : \omega^\omega \to \omega^\omega$ which restricts to a continuous $f : \operatorname{dom} \rho_{\mathcal{X}_0} \to \operatorname{dom} \rho_{\mathcal{X}_1}$ such that for every $\alpha \in \operatorname{dom} \rho_{\mathcal{X}_0}$ we have

$$
l_0 \circ \rho_{\mathcal{X}_0}(\alpha) \leqslant l_1 \circ \rho_{\mathcal{X}_1}(f(\alpha)) \quad \text{in } Q.
$$

Therefore the continuous function $\operatorname{dom} \rho_{\mathcal{X}_0} \to \operatorname{dom} \rho_{\mathcal{X}_1}$ induces a relatively continuous total relation $R : \mathcal{X}_0 \to \mathcal{X}_1$ that witnesses $l_0 \preccurlyeq_{\mathrm{c}} l_1$, i.e. $h(X) \preccurlyeq_{\mathrm{c}} h({}_*X)$, and so h is good. Since h was arbitrary, it follows that $B^{\text{gen}}(Q)$ is BQO. □

When Q is the BQO 2 partially ordered by equality, then $B^{\text{gen}}(2)$ consists of the Borel subsets of Borel representable spaces quasi-ordered by $\preccurlyeq_{\mathrm{W}}$.

Corollary 5.55. *The quasi-order $\preccurlyeq_W$ is well-founded on the Borel subsets of Borel representable spaces.*

Of course assuming the Axiom of Determinacy (AD), the general structural result holds. Namely assuming AD, all subsets of any second countable T_0 space $\mathcal{X}$ are BQO under $\preccurlyeq_{\mathrm{W}}$, and if $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$ are subsets of second countable T_0 spaces, then either $(A, \mathcal{X}) \preccurlyeq_{\mathrm{W}} (B, \mathcal{Y})$, or $(B^{\mathrm{C}}, \mathcal{Y}) \preccurlyeq_{\mathrm{W}} (A, \mathcal{X})$.

These positive results on the structure of the quasi-order $\preccurlyeq_{\mathrm{W}}$ also imply that $\preccurlyeq_{\mathrm{W}}$ often differs with the quasi-order of continuous reducibility $\leqslant_{\mathrm{W}}$. Indeed Schlicht [Sch] showed that in every non 0-dimensional metric space there exists an antichain of the size of the continuum for the continuous reducibility. Using this result and Proposition 5.38, we see that in the separable metrisable case the two notions of reducibility differ as soon as we depart from the zero-dimensional framework.

Corollary 5.56. *Let $\mathcal{X}$ be a metrisable and Borel representable space. Then $\preccurlyeq_W$ and $\leqslant_W$ coincide on subsets of $\mathcal{X}$ if and only if $\mathcal{X}$ is 0-dimensional.*

5.6 The real line and the Scott domain

We now briefly have a closer look at the difference between the continuous reducibility and the reducibility by continuous relations in two important examples.

5.6.1 The Real Line

In [Ike10] (see also [IST]) an embedding from $(\mathcal{P}(\omega), \subseteq_{\mathrm{fin}})$, where $\subseteq_{\mathrm{fin}}$ denotes inclusion modulo finite, i.e. $x \subseteq_{\mathrm{fin}} y \leftrightarrow x \setminus y$ is finite, into the difference of two open sets of the real line ordered by Wadge reducibility is exhibited. We now recall this construction.

Take increasing sequences of real numbers $\langle a_\alpha, b_\alpha \mid \alpha < \omega^\omega \rangle$ indexed by the ordinal ω^ω and $\langle c_n \mid n \geqslant 1 \rangle$ with

$$\begin{aligned}
&a_\alpha < b_\alpha < a_{\alpha+1} && \text{for each } \alpha < \omega^\omega \\
&a_\lambda^- := \sup\{a_\alpha \mid \alpha < \lambda\} < a_\lambda && \text{for each limit } \lambda < \omega^\omega \\
&a_{\omega^n}^- < c_n < a_{\omega^n} && \text{for each } n \in \omega.
\end{aligned}$$

Now for $X \subseteq \omega \setminus \{0\}$ we let

$$D_X = \bigcup_{\alpha < \omega^\omega} [a_\alpha, b_\alpha) \cup \{c_n \mid n \notin X\}.$$

Clearly D_X is a difference of two open sets for all $X \subseteq \omega \setminus \{0\}$.

Theorem 5.57 ([Ike10]). *For every $X, Y \subseteq \omega \setminus \{0\}$,*

$$X \subseteq_{fin} Y \quad \longleftrightarrow \quad D_X \leqslant_W D_Y.$$

By Parovičenko's Theorem [Par93], any poset of size $\aleph_1$ embeds into the partially ordered set $(\mathcal{P}(\omega), \subseteq_{\text{fin}})$, hence there are long infinite descending chains and long antichains for the Wadge reducibility, already among the difference of two open sets of the real line.

As an example, we now give winning strategies witnessing $D_X \preccurlyeq_{\text{W}} D_Y$ for every $X, Y \subseteq \omega \setminus \{0\}$.

Proposition 5.58. *For every $X, Y \subseteq \omega \setminus \{0\}$, we have $D_X \preccurlyeq_W D_Y$.*

Proof. Let $\rho_{\mathbb{R}}$ be the admissible representation of the real line from Example 5.19.

We choose for every $x \in \mathbb{R}$ a particular code via $\rho_{\mathbb{R}}$ by setting $\alpha^x : \omega \to \omega$ to be the increasing enumeration of $\{n \in \omega \mid x \in I_n\}$.

Now fix $X, Y \subseteq \omega \setminus \{0\}$. We describe a winning strategy $\sigma = \sigma_{X,Y}$ for player II in the game $G^{\rho_{\mathbb{R}}}(D_X, D_Y)$. Let J_k be the open interval $(a^-_{\omega^k}, a_{\omega^k})$. And note that we only need to consider positions where Player I has played $(n_0, n_1, \dots, n_j)$ with $\bigcap_{i=0}^{j} I_{n_i}$ is non empty. Let $X \bigtriangleup Y$ denote the symmetric difference of X and Y, i.e.

$$X \bigtriangleup Y = \{x \in \omega \setminus \{0\} \mid \neg(x \in X \leftrightarrow x \in Y)\}.$$

Our winning strategy $\sigma : \omega^\omega \to \omega$ for Player II in $G^{\rho_{\mathbb{R}}}(D_X, D_Y)$ goes as follows:

As long as Player I is in a position where he has played $(n_0, n_1, \cdots, n_j)$ such that $I^j = \bigcap_{i=0}^{j} I_{n_i} \nsubseteq J_k$ for all $k \in X \bigtriangleup Y$, σ consists simply in copying the last move of Player I: n_j. Therefore σ will induce the identity function outside the J_k's for which $k \notin X \bigtriangleup Y$.

Now consider Player I has played $(n_0, n_1, \cdots, n_j)$ such that there exists $k \in X \bigtriangleup Y$ with $I^j = \bigcap_{i=0}^{j} \subseteq J_k$ and let l be the least integer with $I^l = \bigcap_{i=0}^{l} I_{n_i} \subseteq J_k$. We distinguish several cases:

$c_k \in D_Y \setminus D_X$*:* then for σ to be winning for Player II, it must eventually make him play the code of a point outside of D_Y and it cannot be c_k.

Now since $I^l \subseteq J_k$, say $I^l = (r_0, r_1)$, we can for example choose

$$y = \frac{r_0 + \min\{r_1, c_k\}}{2}, \text{ if } r_0 < c_k,$$

$$\text{or} \quad y = \frac{\max\{r_0, c_k\} + r_1}{2}, \text{ if } c_k \leqslant r_0,$$

and play $\alpha^y(j-l)$.

In other words, if Player I enters some J_k with $c_k \in D_Y \setminus D_X$, then σ consists in playing the code of some $y \in J_k$ different from c_k, where y depends on the first position where Player I enters J_k.

$c_k \in D_X \setminus D_Y$ and $c_k \in I^j$: then as long as $c_k \in I^j$, σ must consist in playing as if Player I was going to play c_k, i.e. describe step by step a point belonging to D_Y and it cannot be c_k.

Now since $I^{l-1} \not\subseteq J_k$ (if $l = 0$ set $I^{l-1} = \mathbb{R}$), we choose some $y \in D_Y \cap I_{l-1}$ as follows:

- if $a_{\omega^k} \in I^{l-1}$, then set $y = a_{\omega^k}$,
- otherwise there is a minimal $\beta < \omega^k$ with $a_\beta \in I^{l-1}$, set $y = a_\beta$,

and we play $\alpha^y(j-l)$.

$c_k \in D_X \setminus D_Y$ and $c_k \notin I^j$: then for σ to be winning for Player II, it must eventually make him play the code of a point which is outside of D_Y, but we must be careful to be consistent with what Player II has already played until that point.

Let p be the least integer such that $c_k \notin I^k$. First if $p \leqslant l$, i.e. at the first position where Player I entered J_k we already knew he was not going to play c_k, so we can just copy its last move n_j. Otherwise $l < p$ so $c_k \in I^l$ and we must distinguish two cases:

- if $a_{\omega^k} \in I^{l-1}$, then according to our previous case, at round p, Player II has so far played according to σ:

  $$t = (n_0, n_1, \dots, n_{l-1}, \alpha^{a_{\omega^k}}(0), \alpha^{a_{\omega^k}}(1), \dots, \alpha^{a_{\omega^k}}(p-l-1)).$$

  so $\bigcap_{i=0}^{p-1} I_{t(i)}$ is an open interval (r_0, r_1) with rational endpoints satisfying $r_0 < a_{\omega^k} < r_1$, so we can take

  $$z = \frac{\max\{a^-_{\omega^k}, r_0\} + a_{\omega^k}}{2}$$

  and play $\alpha^z(j-p)$.
- Otherwise according to our previous case, up to round p, the moves of Player II according to σ are

  $$t = (n_0, n_1, \dots, n_{l-1}, \alpha^{a_\beta}(0), \alpha^{a_\beta}(1), \dots, \alpha^{a_\beta}(p-l-1)).$$

where β is the minimal ordinal with $a_\beta \in I^{l-1}$. Again $\bigcap_{i=0}^{p-1} I_{t(i)}$ is an open interval (r_0, r_1) with rational endpoints satisfying $r_0 < a_\beta < r_1$, so we can take

$$z = \frac{\max\{a_\beta^-, r_0\} + a_\beta}{2}$$

where a_β^- stands for $b_{\beta-1}$ if β is successor, and we play $\alpha^z(j-p)$.

It should be clear that σ is a winning strategy for Player II in $G^{\rho_\mathbb{R}}(D_X, D_Y)$. So $D_X \preccurlyeq_\mathrm{W} D_Y$. □

If $X \not\subseteq_\mathrm{fin} Y$, then $X \not\leqslant_\mathrm{W} Y$ by Theorem 5.57 and so the winning strategy for II in $G^{\rho_\mathbb{R}}(D_X, D_Y)$ described in the previous proof induces a continuous $f_{X,Y} : \omega^\omega \to \omega^\omega$. The relation

$$R^{\rho_\mathbb{R}}_{f_{X,Y}}(x,y) \quad \longleftrightarrow \quad \exists \alpha \in \operatorname{dom} \rho_\mathbb{R} \ \big(\rho_\mathbb{R}(\alpha) = x \wedge y = \rho_\mathbb{R}(f_{X,Y}(\alpha))\big)$$

is therefore a relatively continuous relation from $\mathbb{R}$ to $\mathbb{R}$ with no continuous uniformising function. Indeed any function uniformising $R^{\rho_\mathbb{R}}_{f_{X,Y}}$ is a reduction of X to Y and since $D_X \not\leqslant_\mathrm{W} D_Y$ there is no such continuous function.

5.6.2 The Scott Domain

We now give a simple example in the space $\mathcal{P}\omega$ of a case where $\leqslant_\mathrm{W}$ differs from $\preccurlyeq_\mathrm{W}$. Consider $\{\{0\}\}, \{\omega\} \subseteq \mathcal{P}\omega$, we first show that $\{\{0\}\} \not\leqslant_\mathrm{W} \{\omega\}$. To see this, recall that continuous functions on $\mathcal{P}\omega$ are the Scott continuous functions with respect to inclusion, so in particular they are monotone for inclusion. Now since ω is the top element, any monotone map $f : \mathcal{P}\omega \to \mathcal{P}\omega$ with $f(\{0\}) = \omega$ has to send every $X \subseteq \omega$ with $0 \in X$ onto ω too, so that $f^{-1}(\omega) \supseteq O_{\{0\}}$. Therefore no Scott continuous function is a reduction from $\{\{0\}\}$ to $\{\omega\}$.

While we have $\{\{0\}\} \not\leqslant_\mathrm{W} \{\omega\}$, we actually have $\{\{0\}\} \preccurlyeq_\mathrm{W} \{\omega\}$, i.e. there exists a relatively continuous $R : \mathcal{P}\omega \rightrightarrows \mathcal{P}\omega$ such that for all $X, Y \in \mathcal{P}\omega$:

$$X \, R \, Y \quad \longrightarrow \quad (X = \{0\} \leftrightarrow Y = \omega).$$

Clearly any such relation R cannot be uniformised by a Scott continuous function. Indeed such a Scott continuous function would be a reduction between the considered sets, and we know there is none.

We now give such a relation R as a strategy in the game $G^{\rho_{\mathrm{En}}}(\{\{0\}\},\{\omega\})$. Since ρ_{En} is total, we know by Lemma 5.37 that $\{\{0\}\} \preccurlyeq_{\mathrm{W}} \{\omega\}$ if and only if $A \leqslant_W B$ for

$$\begin{aligned} A &= \rho^{-1}(\{\{0\}\}) = \{\alpha \in \omega^\omega \mid \alpha \in 2^\omega \wedge \exists k\ \alpha(k) = 1\} \\ \text{and} \qquad B &= \rho^{-1}(\{\omega\}) = \{\alpha \in \omega^\omega \mid \alpha : \omega \to \omega \text{ is surjective}\}. \end{aligned}$$

A winning strategy for Player II is for example given by the function $\sigma : \omega^{<\omega} \to \omega$ defined by

$$\sigma(s) = \begin{cases} 0 & \text{if } s \in \{0\}^{<\omega} \text{ or } \exists k < |s|\ s_k \neq 0, 1, \\ n & \text{if } s \in 2^\omega \text{ and } n = |s| - \min\{k \mid s_k = 1\}. \end{cases}$$

It is easily seen that this strategy induces a continuous function $f : \omega^\omega \to \omega^\omega$ witnessing the relative continuity of the relation $R : \mathcal{P}\omega \rightrightarrows \mathcal{P}\omega$ given by

$$\begin{aligned} R(\{0\}) &= \{\omega\}, \\ R(X) &= \{\emptyset\} && \text{if } 0 \notin X, \\ R(X) &= \{n \mid n \in \omega\} && \text{if } \{0\} \subset X, \end{aligned}$$

where $n = \{0, \dots, n-1\}$ and $\subset$ denotes strict inclusion.

We close this chapter by mentioning another crucial difference between $\preccurlyeq_{\mathrm{W}}$ and $\leqslant_{\mathrm{W}}$ in relation with complete sets. Recall (e.g. [Kec95]) that $F = \{\alpha \in \omega^\omega \mid \exists n \forall k \geqslant n\ \alpha(k) = 0\}$ is complete for $\mathbf{\Sigma}^0_2(\omega^\omega)$, i.e. $F \in \mathbf{\Sigma}^0_2(\omega^\omega)$ and for every $A \in \mathbf{\Sigma}^0_2(\omega^\omega)$ we have $A \leqslant_{\mathrm{W}} F$. In fact a subset of ω^ω is complete for $\mathbf{\Sigma}^0_2(\omega^\omega)$ if and only if it belongs to $\mathbf{\Sigma}^0_2(\omega^\omega) \setminus \mathbf{\Pi}^0_2(\omega^\omega)$.

The set $\mathcal{P}_{<\infty}(\omega)$ of finite subsets of ω is $\mathbf{\Sigma}^0_2$ in $\mathcal{P}\omega$. It is shown in [BG15, Theorem 5.10] that it is not complete for the Scott continuous reducibility in the class $\mathbf{\Sigma}^0_2(\mathcal{P}\omega)$, i.e. there exists $G \in \mathbf{\Sigma}^0_2(\mathcal{P}\omega)$ such that $G \not\leqslant_{\mathrm{W}} \mathcal{P}_{<\infty}(\omega)$. In contrast the following easy result holds.

Lemma 5.59. *We have* $\mathbf{\Sigma}^0_2(\mathcal{P}\omega) = \{A \subseteq \mathcal{P}\omega \mid A \preccurlyeq_W \mathcal{P}_{<\infty}(\omega)\}$.

Proof. Since $\mathcal{P}_{<\infty}(\omega)$ is $\mathbf{\Sigma}^0_2$ in $\mathcal{P}\omega$, the right to left inclusion follows from Proposition 5.44.

Now for the admissible representation $\rho_{\mathrm{En}} : \omega^\omega \to \mathcal{P}\omega$ from Example 5.21, we have

$$\widetilde{F} = \rho_{\mathrm{En}}^{-1}(\mathcal{P}_{<\infty}(\omega)) = \{\alpha \in \omega^\omega \mid \exists n \forall k\ \alpha(k) \leqslant n\}.$$

Clearly $F \leqslant_{\mathrm{W}} \widetilde{F}$ as the continuous function $f : \omega^\omega \to \omega^\omega$, $f(\alpha)(n) = \mathrm{Card}\{k < n \mid \alpha(k) \neq 0\}$ witnesses. Hence $\widetilde{F}$ is also complete for $\mathbf{\Sigma}^0_2$ in ω^ω. Therefore for any $\mathbf{\Sigma}^0_2$ set $A \subseteq \mathcal{P}\omega$, there is a continuous function $f : \omega^\omega \to \omega^\omega$ which reduces $\rho_{\mathrm{En}}^{-1}(A)$ to $\rho_{\mathrm{En}}^{-1}(\mathcal{P}_{<\infty}(\omega))$, and so $A \preccurlyeq_{\mathrm{W}} \mathcal{P}_{<\infty}(\omega)$. $\square$

We end here our brief exemplification of the difference between the quasi-orders $\leqslant_{\mathrm{W}}$ and $\preccurlyeq_{\mathrm{W}}$ on the real line and the Scott domain. However, the study of the reducibility by relatively continuous relations on these two spaces, and others, yet remains to be realised.

6 Conclusion

The notion of better-quasi-order

One aim throughout this thesis was to understand better the notion of BQO. We have have committed ourselves to motivate as best as we could the definition of BQO in Chapter 2. Nonetheless, we could not help but to notice that the way we arrived to the definition was somehow accidental. We therefore formulated the following problem (see Problem 1):

Problem 4. *Characterise the topological digraphs which can be substituted for* $([\omega]^\infty, \mathsf{S})$ *in the definition of* BQO.

In a first attempt to apprehend this problem, we proved Theorem 2.69 which shows that one can at least substitute a so called generalised shift for the shift map S. At present, a complete solution however seems elusive.

Fronts as uniform spaces

In Chapter 3 we investigated fronts as uniform spaces. We believe that our results, especially Theorems 3.24 and 3.25, entertain the relevance of this approach and we hope that these ideas can be applied in other contexts where fronts are used. In any case, these techniques allow for a smooth proof of Pouzet's conjecture as presented in Chapter 4.

Unravel the notion of better-quasi-order

One obstacle in understanding the notion of BQO may lie in the difficulty to appreciate the gap between being WQO and being BQO. In this regard our Theorem 4.40 (Pouzet's conjecture) provides a first step in understanding the notion of BQO. Notice that it provides us with a process which can be iterated. Suppose Q is WQO, and consider $\mathrm{Id}^*(Q)$. If $\mathrm{Id}^*(Q)$ is not WQO, then Q is not BQO. But assume now that $\mathrm{Id}^*(Q)$ is WQO, we can apply Theorem 4.40 once again and can consider $\mathrm{Id}^*(\mathrm{Id}^*(Q)) = \mathrm{Id}^2(Q)$. If $\mathrm{Id}^2(Q)$ is not WQO, then Q is not BQO, otherwise we can consider $\mathrm{Id}^*(\mathrm{Id}^2(Q)) = \mathrm{Id}^3(Q)$, so on and so forth. There are however examples of WQOs such that $\mathrm{Id}^n(Q)$ is WQO for ever natural number n, while Q admits a bad super-sequence from the Schreier barrier. We believe that finding a way to go on unravelling a WQO into the transfinite could give new insights into the notion of BQO.

As an example of a related problem we cannot answer, we would like to mention the following challenging question (see [Pou78]):

Problem 5 (R. Bonnet)**.** *Is every* WQO *poset a countable union of* BQO *posets?*

Better-quasi-ordering the subsets of a topological space

In Chapter 5 we have defined a quasi-order on the subsets of an arbitrary second countable topological T_0 space by considering reductions by relatively continuous relations. We think it conveys a natural idea of relative complexity and we have proved that it refines the classical Borel and Hausdorff-Kuratowski hierarchies. Moreover we showed that, contrary to the quasi-order of reducibility by continuous functions, it is BQO on the Borel subsets of every space belonging to the very large class of Borel representable spaces.

However, this quasi-order has not yet been really studied on alternative spaces, not even on Polish spaces. To give an indication of what could be expected, it is worth mentioning that Duparc and Fournier [DF] have studied the quasi-order of reducibility by relatively continuous relations on the space $\omega^{\leqslant\omega}$ of finite or infinite sequences of natural numbers, equipped with the *prefix topology*[1]. They showed that for every Borel *non-self-dual* subset A of ω^ω – i.e. a Borel subset A of ω^ω such that $A \not\leqslant_{\mathrm{W}} A^{\complement}$ – there exists a Borel subset A' of $\omega^{\leqslant\omega}$ such that $A \preccurlyeq_{\mathrm{W}} A'$ and $A' \preccurlyeq_{\mathrm{W}} A$, and moreover every Borel subset A' of $\omega^{\leqslant\omega}$ appears in this way.

This result motivates the following general question:

Problem 6. *Let $\mathcal{X}$ be an uncountable quasi-Polish space. Is it true that for every non-self-dual Borel subset A of ω^ω there exists a subset A' of $\mathcal{X}$ with $A \preccurlyeq_W A'$ and $A' \preccurlyeq_W A$?*

In a different but related direction, we recall that one very satisfactory fact about the Wadge quasi-order on the Baire space is crystallised in the following observation:

[1] Namely the Scott topology associated with the prefix relation, a base for which is given by the sets of the form $\{u \in \omega^{\leqslant\omega} \mid s \sqsubseteq u\}$ for some $s \in \omega^{<\omega}$.

> Thus we have come to a full circle – non-self-dual pointclasses considered by early descriptive set theorists were defined in terms of (explicit) operations, and assuming AD every non-self-dual pointclass is defined in terms of operations on open sets.
>
> Andretta and Louveau [AL, p.5]

Now if $\mathcal{X}$ is a quasi-Polish space, let us say that a family Γ of subsets of $\mathcal{X}$ is a *non-self-dual initial segment* for $\preccurlyeq_{\mathrm{W}}$ if Γ is not closed under complementation and, $A \preccurlyeq_{\mathrm{W}} B$ and $B \in \Gamma$ implies $A \in \Gamma$. We believe that the following question is worth studying.

Problem 7. *Assume AD and let $\mathcal{X}$ be a quasi-Polish space. Is every non-self-dual initial segment for $\preccurlyeq_W$ defined in terms of operations on open sets?*

List of Symbols

Below are listed the main symbols and abbreviations used throughout the thesis together with a short description and the numbers of the pages where they are introduced.

$X/s,\ X/n$	$\{k \in X \mid k > \max s\},\ \{k \in X \mid k > n\}$
$[X]^k$	$\{s \subseteq X \mid \lvert s\rvert = k\}$
$S^{\uparrow}$	the set of upper bounds of S
$\bigcup F$	$\{n \mid \exists s \in F\ n \in s\}$
$\downarrow S,\ \downarrow q$	$\{p \mid \exists s \in S\ p \leqslant s\},\ \{p \mid p \leqslant q\}$
$\hat{P}$	the profinite completion of the po P
$[X]^{\infty}$	the infinite subsets of X
ω^{op}	the opposite of the po $(\omega, \leqslant)$
$\sqsubseteq,\ \sqsubset$	$s \sqsubset t \leftrightarrow \exists k \in t\ s = \{n \in t \mid n < k\}$
${}_*X$	the shift $X \setminus \{\min X\}$
$\lhd$	$s \lhd t \leftrightarrow \exists X \in [\omega]^{\infty}\,(s \sqsubset X \wedge t \sqsubset {}_*X)$
$p \mid q$	$p \nleqslant q$ and $q \nleqslant p$
$\lvert s\rvert$	the cardinality of s
$B(Q)$	the qo of Borel Q-labelling functions of 0-dim Luzin spaces
$B^{\mathrm{gen}}(Q)$	the qo of Borel Q-labelling functions of Borel representable spaces
BQO	better-quasi-order
$\mathcal{D}(Q)$	downsets of Q with inclusion
$\mathcal{D}_{\mathrm{fb}}(Q)$	finitely bounded downsets of Q with inclusion
$\mathbf{D}(L)$	the Priestley dual of the lattice L
$\boldsymbol{\Delta}^0_{\alpha}$	the ambiguous class α
$D_{\xi}(\boldsymbol{\Sigma}^0_{\alpha})$	the class of ξ-differences of $\boldsymbol{\Sigma}^0_{\alpha}$
$\mathbf{E}(\mathcal{X})$	the lattice dual to the Priestley space $\mathcal{X}$
G_{V^*}	the game defining the qo on $V^*(Q)$
$G(l_0, l_1)$	the game defining the qo on $\mathfrak{L}^B(Q)$
$G^{\rho_{\mathcal{X}},\rho_{\mathcal{Y}}}(A, B)$	the reduction game relatively to the representations
$H^*_{\omega_1}Q$	qo of hereditarily countable non-emptys sets over Q
$\mathrm{Id}(P)$	the ideal completion of the po P, the ideal space of the WQO P
$\mathrm{Id}^*(P)$	the po of non principal ideals of P

$\mathfrak{L}^B(Q)$	the qo of Borel Q-labelling functions of Borel subsets of ω^ω
$\leqslant^{\mathrm{rep}}$	the quasi-order on partial functions $f :\subseteq \omega^\omega \to \mathcal{X}$
$\mathrm{CId}(P)$	the Cauchy ideal completion of P
$V^*(Q)$	qo of non-empty sets over Q
$\mathcal{P}_{<\aleph_1}(Q)$	qo of countable subsets of Q
$\mathcal{P}(Q)$	powerset of the qo Q
$\mathcal{S}$	the Schreier barrier
$\mathrm{Im}\, f$	the image or range of the function f
$\mathbf{\Pi}^0_\alpha$	the multiplicative class α
$\mathcal{P}\omega$	the Scott domain
$\mathfrak{R}$	Rado's partial order
ρ_{En}	an admissible representation of the Scott domain
$\rho_{<\infty}$	another admissible representation of the Scott domain
$\rho_{\mathbb{R}}$	an admissible representation of the real line
$\mathrm{seq}_{n\in X} F_n$	$\{\{n\} \cup s \mid n \in X \text{ and } s \in F_n\}$
$\mathbf{\Sigma}^0_\alpha$	the additive class α
$\sigma_{\mathbb{R}}$	an admissible representation of the real line based on Cauchy sequences
$\mathrm{TailLat}(P)$	the lattice lattice of the po P
$\preccurlyeq_{\mathrm{W}}$	reducibility by relatively continuous relations
WQO	well-quasi-order

Index

Below are listed the main definitions and theorems appearing in this thesis together with the numbers of the pages where they are introduced and discussed.

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